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Stability of Steel Structures

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STABILITY OF STEEL STRUCTURES

BY

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THESIS

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ABSTRACT

In the design of an elastic structure, a major consideration is the overall stability of the system under axial load or a combination of an axial load and bending moment. In first-order analysis the effects of axial forces on the stiffness formulation and their influence on deformations of the members are usually neglected. If the axial forces are sufficiently large, a better indication of the distribution of the forces and moments in the structure is obtained by using a second-order analysis.

In this type of analysis the moment equilibrium equations are based on the deformed shape of the structure. The secondary moments produced by axial forces acting through the deformed structure are considered in the computation of stiffness term. The secondary moments are called P-Delta effects. In this paper, by using a finite element computer program, the effects of axial loads on a plane frame structure are investigated. The method of analysis, based on the critical values of applied lateral forces on the elastic stability of the frame, will be referred to in this paper as a second order analysis. Example problems are included to serve to document this analytical approach.

DEDICATION

To my parents, whose love and understanding made this possible.

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LIST OF SYMBOLS

The following symbols are used in this paper:

A	=	cross section area
C_m	=	equivalent moment factor
E	=	modulus of elasticity
f_a	=	computed axial stress
f_{bx}	=	computed bending stress at member end about x axis
f_{by}	=	computed bending stress at member end about y axis
F_{bx}	=	bending stress permitted of bending moment about x axis existed
F_{by}	=	bending stress permitted of bending moment about y axis existed
F_a	=	axial stress permitted if axial force existed
F_e	=	$14900/(KL/r_x)^2$, in which KL/r_x is the effective slenderness ratio in plane of bending
F_{xj}	=	axial load at node j
F_{xk}	=	axial load at node k
F_y	=	minimum yield stress of the type of steel being used
F_{yj}	=	vertical load at node j
F_{yk}	=	vertical load at node k
I	=	moment of inertia
K_{11}	=	restraining moment at end 1
K_{12}	=	shear force at end 1
K_{21}	=	restraining moment at end 2

K_{22} = shear force at end 2

L_i = length of member i

M_j = moment at node j

M_k = moment at node k

P_a = axial force for member

P_E = buckling load for member

P.E. = potential energy

S.E. = strain energy

S_i = stiffness matrix for element i

T_i = coefficient relating the effect of axial force on stiffness

TK_{11} = moment coefficient for fixed end action at node 1

TK_{12} = shear coefficient for fixed end action at node 1

TK_{21} = moment coefficient for fixed end action at node 2

TK_{22} = shear coefficient for fixed end action at node 2

$\{U_i\}$ = displacement matrix

$\{U_i\}^T$ = transpose of displacement matrix

U_{xj} = displacement at x direction for node j

U_{xk} = displacement at x direction for node k

U_{yj} = displacement at y direction for node j

U_{yk} = displacement at y direction for node k

W = concentrated load

$\emptyset_n$ = shape function

θ_j = rotation at node j

θ_k = rotation at node k

CHAPTER I

INTRODUCTION

In general, the analysis of elastic structural systems are based on first order analysis. In this type of analysis, the distribution of forces and moments throughout the structure are developed by using undeformed geometry of the structure. Therefore, the effect of the axial forces on deformed structure are completely ignored. If the magnitudes of the axial forces are sufficiently large, their influence on deformed members will produce additional moments which can be noticably large and will intend to decrease the overall strength of the structure. Thus, a better indication of the distribution of the forces and moments in the structure is obtained by using second order analysis. In the latter, the moment equilibrium equations are based on the deformed shape of the structure and the secondary moments produced by axial forces acting through the deformed structure are considered in the computation of member stiffness term.

Today, the design of most multistory plane frame structures is based on the allowable stress technique. This type of analysis uses an elastic first-order procedure. The bending moments calculated by a first order approach are used directly in the design of the girders; the moments in the column are then calculated, employing the moment amplification equations.[2] For this type of design procedure a factor is used in the moment amplification equation to adjust for the absence of secondary moments in the design. In the next section, we will show the allowable stress technique that is the first-order procedure in more detail. Later an analysis of a plane frame with members subjected to lateral forces combined with bending moments will be made. The method of analysis is based on the critical values of applied lateral forces and their influence on the elastic stability of frame.

It is clear that a compressive axial load reduces a member's overall effective bending resistance and thereby causes a greater deformation. Tensile forces, on the other hand, reduce deformations. In both cases, it is possible to determine the interrelationship among the loading parameters, the unit bending stiffness coefficient and the resulting deformations for a constant value of axial load. Then, assuming that buckling does not occur, one can define the stiffness coefficient and flexibility including the effects of axial load.[1] The derivations and application of the stability approach will be shown in Appendix I.

The analysis is not straightforward due to the fact that the axial forces in the members are related to the joint displacements. Therefore, an analysis must be conducted in a cyclic fashion. In the first cycle of analysis, the stiffness matrix of a member is based on a first-order analysis. In the second cycle the lateral forces in the members resulting from the first cycle are used in determining the modified member stiffness matrix and also the modified fixed-end moments and shears; in this manner, new values are obtained for the lateral forces.

After each solution is completed, the results are compared with the previous solution. If the difference is small, the cyclic process is terminated and the last solution is taken as being correct.

The finite element program of frame analysis[6] has been modified to accomplish the objective of second-order analysis. This program is written in fortran language and is listed under Appendix II. Example problems are included to illustrate the accuracy of the results. These analyses were performed with a 64 bit (cyber 176 mainframe) computer and are presented in a later chapter of this report.

CHAPTER II

REVIEW OF CURRENT DESIGN PRACTICE

At the present time, most steel structures are designed using the allowable stress technique. In this technique, the structure is analyzed under various loading combinations, using an elastic first order procedure. The bending moments are used directly in the design of the girder, while the moments and forces in the columns are checked against the moment amplification equations to arrive at suitable column sections. In this design procedure, terms in the interaction equation are adjusted to compensate for the neglect of secondary moment effects on individual moments.[2] Members subjected to both axial compression and bending stresses will be proportioned at all points along their length to satisfy the following requirements:

$$\frac{f_a}{F_a} + \frac{C_{mx}\, f_{bx}}{\left(1-\frac{f_a}{F'_{ex}}\right)F_{bx}} + \frac{C_{my}\, f_{by}}{\left(1-\frac{f_a}{F'_{ey}}\right)F_{by}} \leq 1.0 \qquad (1a)$$

or:

$$\frac{f_a}{0.60F_y} + \frac{f_{bx}}{F_{bx}} + \frac{f_{by}}{F_{by}} \leq 1.0 \qquad (1b)$$

where:

f_a = computed axial stress

F_a = axial stress permitted if axial force existed

f_{bx} = computed bending stress at member end about x axis

f_{by} = computed bending stress at member end about y axis

F_{bx} = bending stress permitted of bending moment about x axis existed

F_{by} = bending stress permitted of bending moment about y axis existed

F_e = $14900/(KL/r_x)^2$, in which KL/r_x is the effective slenderness ratio in plane of bending

F_y = minimum yield stress of the type of steel being used

In Equations (1a) and (1b), the subscripts x and y combined with subscripts b, m and e indicate the axis of bending about which a particular stress or design property applies.[2]

The expressions

$$\left(\frac{1}{1 - \frac{f_a}{F'_{ex}}}\right) \quad \text{and} \quad \left(\frac{1}{1 - \frac{f_a}{F'_{ey}}}\right)$$

in Equation (1a) are called the amplification factors. It is intended through their existence to account for secondary moments produced by the lateral load, acting on the deformed member. Depending on the shape of the applied moment diagram (the critical location and magnitude of the induced eccentricity), the amplification factor may overestimate the extent of the secondary moment. To allow for this, the amplification factor is modified, as required, by a moment factor C_m. When bending occurs about the x and y axis, the bending stress calculated about each of these axes is adjusted by the value of C_m and F'_e corresponding to the distribution and the slenderness ratio in its plane of bending. The bending stress is then taken as a fraction of that permitted for bending about that axis, with due regard to the unbraced length of compression flange when this is a factor.

When the computed axial stress is no greater than 15 percent of the allowable axial stress, the influence of

$$\frac{C_m}{\left(1 - \frac{f_a}{F_e}\right)}$$

is generally small and may be neglected, and Equation (1a) could reduce to:

$$\frac{f_a}{F_a} + \frac{f_{bx}}{F_{bx}} + \frac{f_{by}}{F_{by}} \leq 1.0 \qquad (2)$$

However, its use in Equation (1a) is not intended to permit a value of f_b greater than F_b when the value of C_m and f_a are both small.[2] Depending upon the unbraced length of a member in the plane of bending, the combined stress computed at one or both ends of this length may exceed the combined stress at all intermediate points where lateral displacement is created by the applied moments.

The limiting value of the combined stress ratios in this case is pointed out in Equation (1b).

C_m in Equation (1a) will increase if the primary bending moment is not uniform over the member length. The strength of the column will also be increased, and because C_m is directly related to the strength of the column, it will also increase. So the equivalent moment factor C_m would be:[2]

$$C_m = 0.60 - 0.40 \frac{M_1}{M_2} \quad \text{with} \quad C_m \geq 0.40$$

where $\frac{M_1}{M_2}$ is the ratio of smaller to larger moments at the ends of that portion of the member unbraced in the plane of bending under consideration.[2]

The ratio of $\frac{M_1}{M_2}$ is positive when the member is bent in a reversed curvature, and it has a negative magnitude when bent in a single curvature. The value of C_m can differ for the following conditions:[2]

1. For compression members in frames subject to joint translation (sideway). $C_m = 0.85$.
2. For compression members in frames braced against joint translation in the plane of loading and subjected to transverse loading between their supports, the value of C_m may be determined by a rational analysis. However, instead of such an analysis, the following values may be used:

   a) For members with both ends restrained, $C_m = 0.85$

   b) For members with both ends not restrained, $C_m = 1.0$

Furthermore, considerable attention has also been given to another technique called the effective column length factor. It is believed in using the effective column length factor, K, that an adequate allowance is made for the secondary moment. The value of K is obtained by constructing a straight line between G_A and G_B from the alignment chart[2] shown in Figure 1.

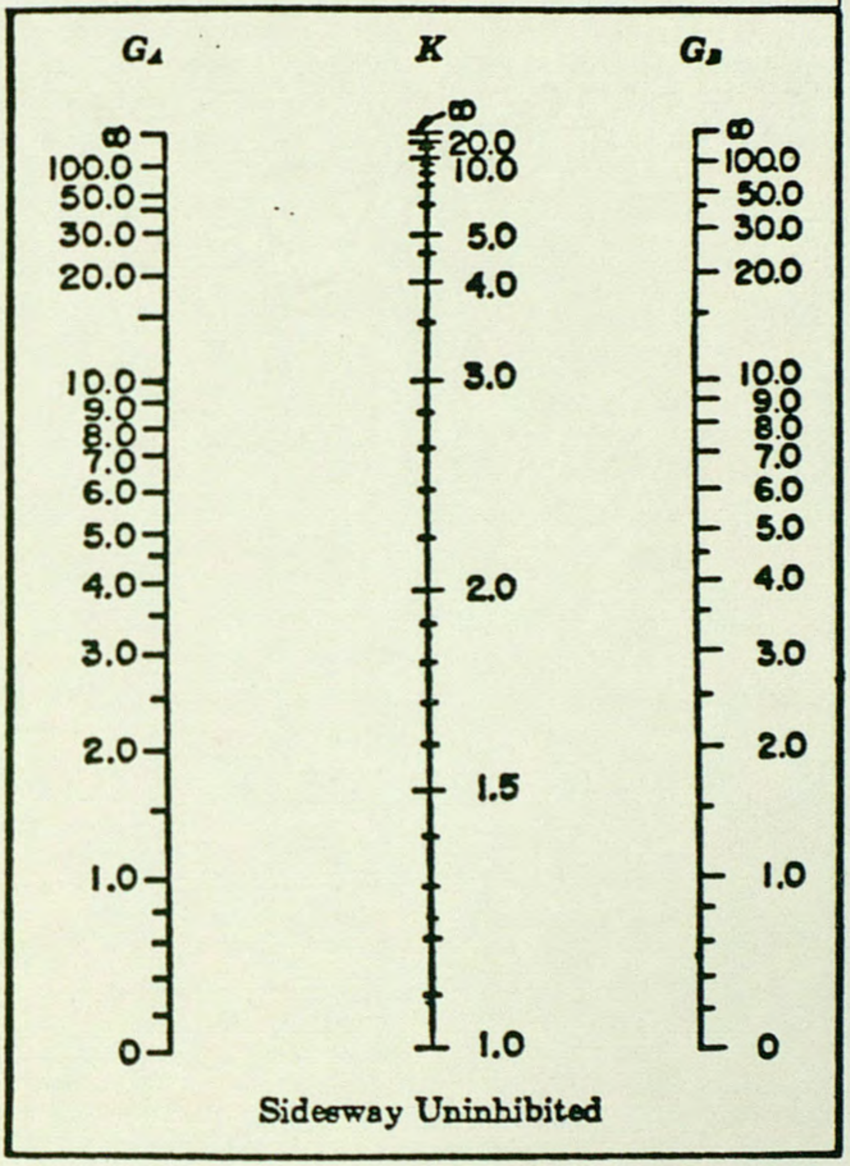


Figure 1. Alignment chart for effective length of column in continuous frames. (Ref. 2)

G_A and G_B are determined from $G = \dfrac{\Sigma \frac{I_c}{L_c}}{\Sigma \frac{I_g}{L_g}}$

The subscripts A and B represent the two ends of the column section and $\sum$ indicates a summation of all members rigidly connected to that join. I_c and L_c are the moment of inertia and unbraced length of the column selection and I_g and L_g the corresponding unbraced length or other restraining member respectively. Although the present K-factor approach is not always satisfactory for columns, neither from a safety or economic point of view, its concept can be easily demonstrated by considering the three frames shown in Figure 2.

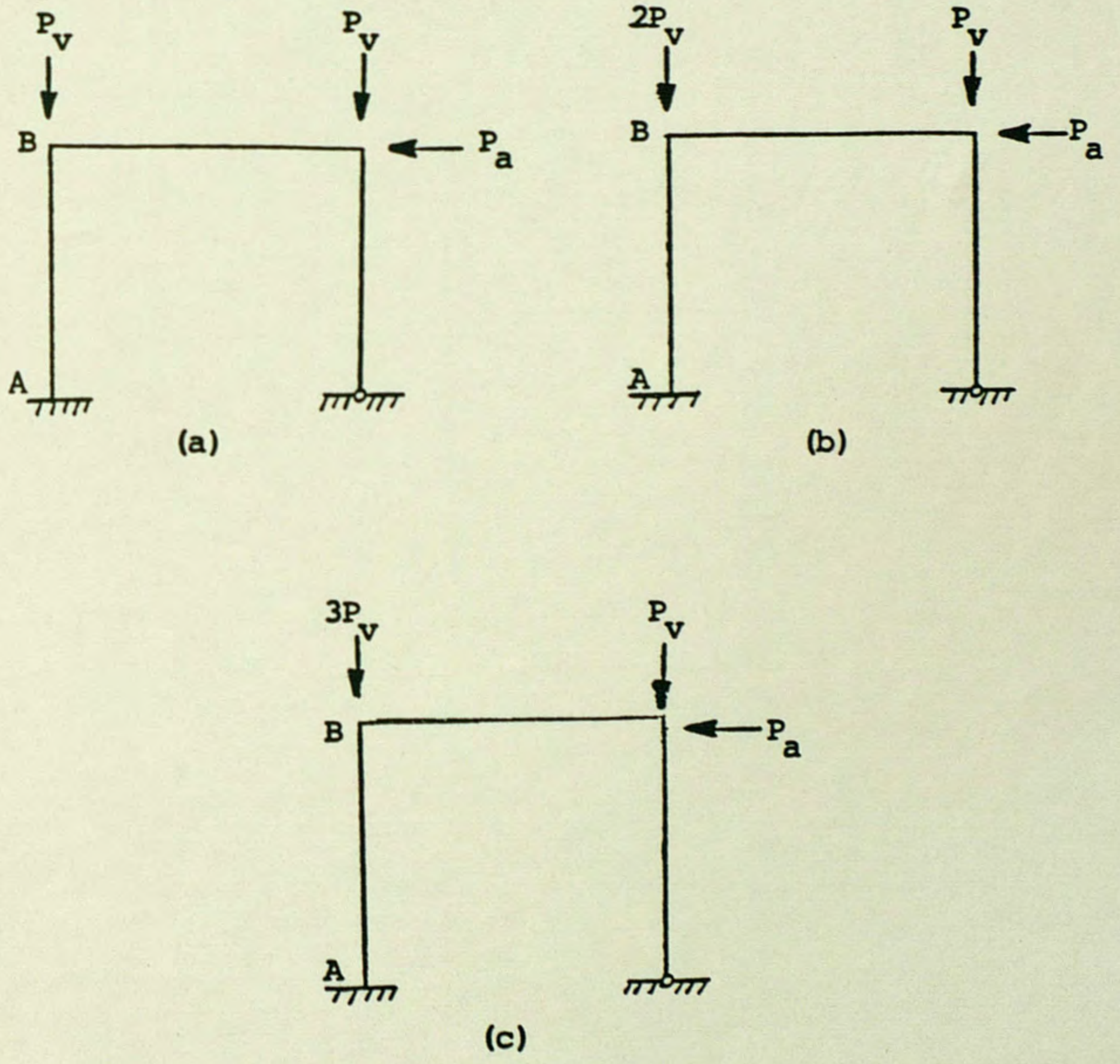


Figure 2. Frame with three different axial loadings.

Column AB, which can resist overturning moments and lateral loads (as well as instability effects), has the same cross section in the three frames shown. Since K, as obtained from the alignment chart, is the same in all three cases, it is clear that if the design obtained is adequate for frame (a), it must then be under designed for frame (c) in which the secondary moment effect is more significant. Or if it is adequate for frame (c), it must be over designed for frame (a), in which the instability effect is smaller. Ideally, there should be three different designs for the three frames of Figure 2, each reflecting the degree of severity of the secondary effect that exists.[3]

Although the K factor provides a means of accounting for the secondary effect in a column, it cannot ensure overall frame stability since it cannot control the instability effect in the compressive beam column. The beam in frame (a), for example, should not only be dimensioned based on a bending moment to lateral load P_a, but also it should consider the effect of the vertical force on sway deformation. In general, because no provision similar to the K factor has been made for the beams, these members will be under designed and may well lead to overall frame distress. An essential step for an improved design, therefore, must be the direct consideration of the secondary overturning moment in the beams.[3]

CHAPTER III

METHOD OF ANALYSIS

The finite element technique has been found to be a very powerful tool for solving structural problems. The concept of the finite element method consists of subdividing the structure into many subregions which are referred to as elements. The intersections of the sides of the elements are called nodes or grids. The nodes are used to specify the geometry of individual elements. There is no fixed rule that can be used to determine the number of nodes and elements needed in a finite element mesh. In general, the finer the mesh, the better will be the numerical solution that is obtained. As more computer time is needed as the number of nodes and elements increase, the process eventually leads to excessive costs. Therefore, at the start of an analysis, it is up to engineering judgement and previous experience to find the coarsest possible mesh, which will achieve a numerical solution within the desired precision.

Once a finite element mesh is selected, nodes and elements are numbered for reference purposes in the remaining steps. The nodal points of the element provide mathematical expression for defining the shape as an unknown quantity over the domain of the element. A set of mathematical functions such as polynominals or trigonometric series can be used for this purpose in the finite element formulation.[4]

If U is denoted as the unknown quantity for element , then the polynominal series can be written as:

$$U = \phi_1 U_1 + \phi_2 U_2 + \ldots\ldots\ldots + \phi_n U_n \tag{3}$$

where U_1, U_2, ..., U_n are the unknown values at the nodal points and ϕ_1, ϕ_2, ..., ϕ_n are the interpolation functions.

For example, in the case of displacement, these interpolation functions represent the deformed shape of an element under given loading conditions. Also, U_1, U_2, ..., U_n are the unknown values of displacements at each node, that are also called degrees of freedom.

The interpolation functions must also satisfy the compatibility requirements for displacements along the boundaries with adjacent elements. There are many different types of approximate methods. The two most important methods for engineering applications are the Rayleigh-Ritz and Galerkin method. The concept of minimum potential energy of Rayleigh-Ritz applied for computing the stiffness matrix is used in the Finite Element Program. The potential energy is obtained as the summation of the internal strain energy minus the potential of the external loads.[4] The equilibrium relation for elements consists of the stiffness matrix $[S_i]$, the displacement vector $\{U_i\}$, and the nodal force vector $\{F_i\}$. These relations are defined as a set of simultaneous linear equations, which evolve from a process of minimizing the potential energy in accordance with the associated conditions.

Potential Energy (P.E.) = Strain Energy (S.E.) - Work

After various steps of derivation are completed, the potential energy in effect for element i can be written as:

$$(P.E.)_i = \frac{1}{2}\{U_i\}^T [S_i]\{U_i\} - \{U_i\}^T \{F_i\}$$

The potential energy (P.E.) is then minimized by taking its derivative with respect to the various displacements and setting these qualities to zero.

$$\frac{\partial (P.E.)_i}{\partial U} = 0 \qquad i = 1, 2 \ldots., m$$

or:

$$\frac{\partial}{\partial U_i} [\frac{1}{2} \{U_i\}^T [S_i] \{U_i\} - \{U_i\}^T \{F_i\}] = 0$$

Thus:

$$[S_i] \{U_i\} - \{F_i\} = 0$$

or:

$$[S_i] \{U_i\} = \{F_i\} \qquad (4)$$

For a simple one dimensional, two noded element i shown in Figure 3, the displacement and force vector can be written knowing that each node includes two translational and one rotational degrees of freedom.

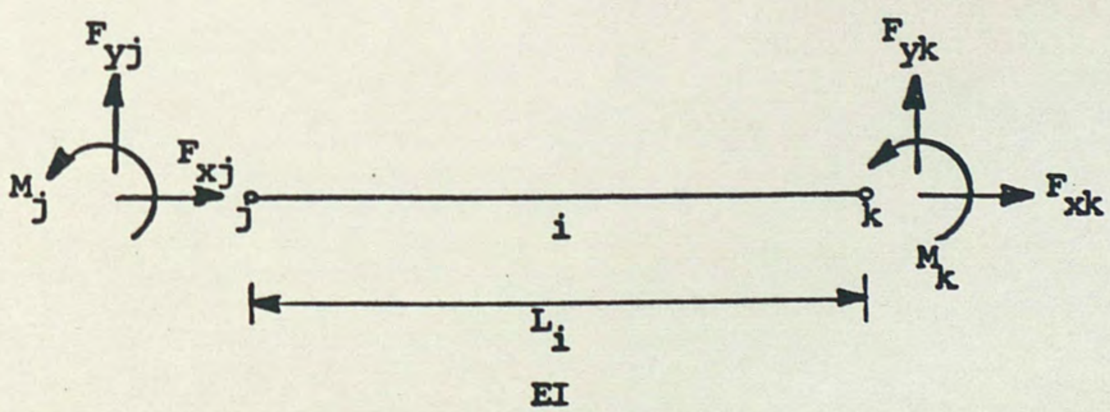


Figure 3. Free body diagram of element i.

The translational and rotational shape functions for the element shown in Figure 3 are:

$$\phi_1 = [1 - 3(\frac{x}{L_i})^2 + 2(\frac{x}{L_i})^3]$$

$$\phi_2 = [x - 2\frac{x^2}{L_i} + \frac{x^3}{L_i^2}]$$

and

$$\phi_3 = [3(\frac{x}{L_i})^2 - 2(\frac{x}{L_i})^3]$$

$$\phi_4 = [-\frac{x^2}{L_i} + \frac{x^3}{L_i^2}]$$

$$\{F_i\} = \begin{Bmatrix} F_{xj} \\ F_{yj} \\ M_j \\ F_{xk} \\ F_{yk} \\ M_k \end{Bmatrix} \quad \text{and} \quad \{U_i\} = \begin{Bmatrix} U_{xj} \\ U_{yj} \\ \theta_j \\ U_{xk} \\ U_{yk} \\ \theta_k \end{Bmatrix}$$

And stiffness matrix can be obtained from:

$$S_{jk} = \int_0^L EI \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_k}{\partial x^2} dx$$

where: E and I represent the modulus of elasticity and moment of inertia of element i respectively;

ϕ_j and ϕ_k represent the interpolation or shape functions at node j and k. By using the above equation, the stiffness matrix can be easily derived and will lead to:

$$S_i = \frac{E}{L} \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I}{L^2} T_1 & \frac{6I}{L} T_2 & 0 & -\frac{12I}{L^2} T_1 & \frac{6I}{L} T_2 \\ 0 & \frac{6I}{L} T_2 & 4I\, T_3 & 0 & -\frac{6I}{L} T_2 & 2I\, T_4 \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & -\frac{12I}{L^2} T_1 & -\frac{6I}{L} T_2 & 0 & \frac{12I}{L^2} T_1 & -\frac{6I}{L} T_2 \\ 0 & \frac{6I}{L} T_2 & 2I\, T_4 & 0 & -\frac{6I}{L} T_2 & 4I\, T_3 \end{bmatrix} \qquad (5)$$

where $T_1 = T_2 = T_3 = T_4 = 1$, in a first-order analysis.

The total potential energy of the entire structure is the summation of potential energy of each individual element. Thus,

$$\sum_{i=1}^{m} \text{P.E.} = \sum_{i=1}^{m} \frac{1}{2} \{U_i\}^T [S_i] \{U_i\} - \sum_{i=1}^{m} \{U_i\}^T \{F_i\} \qquad (6)$$

where m is the total number of elements of the structure.

A solution cannot be obtained until the geometric boundary conditions are taken into account. Boundary conditions are the physical constraints or supports that must exist so that the structure can stand in space uniquely.

After the boundary conditions are introduced, the above set of equations are solved for the unknown displacements. Equation (5) shows the general stiffness matrix of a member. For the first cycle, the values of T_1, T_2, T_3, and T_4 which are the coefficients relating the effect of axial force on bending stiffness terms, are one. And the axial stiffness is assumed to be unaffected by moments.

To account for secondary moments, the stiffness matrix has been modified based on the critical values of applied lateral forces on each individual member. Therefore, by using certain stability functions which are related to the equilibrium conditions of a deformed structure, the stiffness of each member can be modified to present its true strength.

The analysis is conducted in a cyclic fashion. After the first cycle is completed, the lateral forces obtained from the first cycle are used to evaluate the T_i coefficients of each member throughout the entire structure for the next cycle. The cyclic process continues until two successive analyses yield approximately the same results.[5] Values of T_i are shown in Table 1.

TABLE 1

COEFFICIENTS RELATING THE EFFECT OF AXIAL FORCE ON STIFFNESS (Ref. 5)

	CONDITION OF AXIAL FORCE		
Coefficient	Compression	Zero	Tension
T_1	$(KL)^3$ Sin KL/12 fc	1	$(KL)^3$ Sinh KL/12 ft
T_2	$(KL)^2$ (1-Cos KL)/6 fc	1	$(KL)^2$ (Cosh KL-1)/6 ft
T_3	KL(Sin KL-KL Cos KL)/4 fc	1	KL(KL Cosh KL-Sinh KL)/4 ft
T_4	KL(KL-Sin KL)/2 fc	1	KL(Sinh KL-KL)/2 ft

where: $fc = 2 - 2 \text{ Cos } KL - KL \text{ Sin } KL$

$ft = 2 - 2 \text{ Cosh } KL + KL \text{ Sinh } KL$

$$KL = \pi \sqrt{\frac{P_a}{P_E}}$$

P_E is the Euler Buckling load for a pin-ended column with no transverse load and is evaluated:

$$P_E = \frac{\pi^2 EI}{L^2}$$

If the axial force in any member equals the load where buckling occurs, a singular matrix occurs and results are not obtained.

Since fixed-end action values are affected by the axial load, therefore fixed-end action as well as stiffness in members should be recomputed for each cycle.[5] Table 2 shows the value of coefficients for evaluating the fixed-end action for the concentrated load members shown in Figure 4.

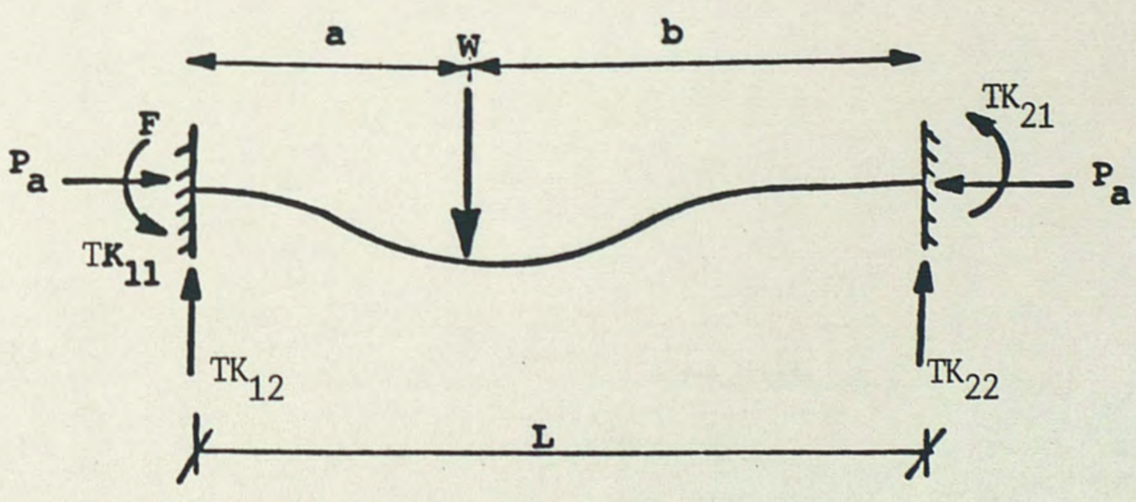


Figure 4. Free body diagram, beam with concentrated lateral load.

TABLE 2

COEFFICIENTS RELATING THE EFFECT OF AXIAL FORCE
ON FIXED END ACTION (Ref. 5)

Coefficient	CONDITION OF AXIAL FORCE		
	Compression	Zero	Tension
TK_{12}	$\frac{1}{fc}(\text{Cos KL} - \text{Cos Ka} + \text{Cos Kb} + \text{Kb Sin KL} - 1)$	$\frac{-b^2(3a+b)}{L^3}$	$\frac{1}{ft}(\text{Cosh KL} - \text{Cosh Ka} + \text{Cosh Kb} + \text{Kb Sinh KL} - 1)$
TK_{11}	$\frac{1}{Kfc}(\text{Sin KL} - \text{Sin Ka} - \text{Sin Kb} - \text{Kb Cos KL} + \text{KL Cos Kb} - \text{Ka})$	$\frac{-ab^2}{L^2}$	$\frac{1}{Kft}(\text{Sinh KL} - \text{Sinh Ka} - \text{Sinh Kb} - \text{Kb Cosh KL} + \text{KL Cosh Kb} - \text{Kb})$
TK_{22}	$\frac{1}{fc}(\text{Cos KL} - \text{Cos Kb} + \text{Cos Ka} + \text{Ka Sin KL} - 1)$	$\frac{-a^2(a+3b)}{L^3}$	$\frac{1}{ft}(\text{Cosh KL} - \text{Cosh Kb} + \text{Cosh Ka} - \text{Ka Sinh KL} - 1)$
TK_{21}	$\frac{1}{Kfc}(-\text{Sin KL} + \text{Sin Kb} + \text{Sin Ka} + \text{Ka Cos KL} - \text{KL Cos Ka} + \text{Kb})$	$\frac{a^2b}{L^2}$	$\frac{1}{Kft}(-\text{Sinh KL} + \text{Sinh Kb} + \text{Sinh Ka} + \text{Ka Cosh KL} - \text{KL Cosh Ka} + \text{Kb})$

where: $fc = 2 - 2\ \text{Cos KL} - \text{KL Sin KL}$

$ft = 2 - 2\ \text{Cosh KL} + \text{KL Sinh KL}$

$K = \sqrt{\frac{P_a}{EI}}$

In case of uniform loaded member, similar terminology applied and the uniform member sub-divided into small segments and each segment was treated as a member with concentrated load at the center of each portion.

The cyclic fashion of analysis described in this chapter may be used to evaluate the buckling load for a frame. The critical load on the member may be found by increasing the load in increments until the stiffness matrix becomes singular. This singularity is the basis for obtaining the magnitude of loading which causes elastic instability in the fundamental buckling mode.[5]

CHAPTER IV

PROGRAM IMPLEMENTATION

This chapter will briefly discuss the program's operations and capabilities.

The program is capable of solving two-dimensional frame problems with concentrated and uniform load conditions. The computations are conducted in a cyclic fashion. After each solution is completed, the axial force of each member is compared with the previous solution. If the absolute difference of the axial force between two successive cycles is less than .0001, the computation process is terminated and the last solution is taken as being the final result.

The main program will initialize the program's parameters and then call on a set of six different subroutines. The sequence of operations is shown in Figure 5, where:

NRMX = Row dimension for the stiffness matrix

MCMX = Column dimension for the stiffness matrix or maximum half band width allowed

NDF = Number of degrees of freedom per node

NNE = Number of node per element

NDFEL = Total number of degrees of freedom for one element

FORC((I-1)*6+1) = Axial force in local coordinate system from previous cycle

PF((I-1)*6+1) = Axial force in local coordinate system

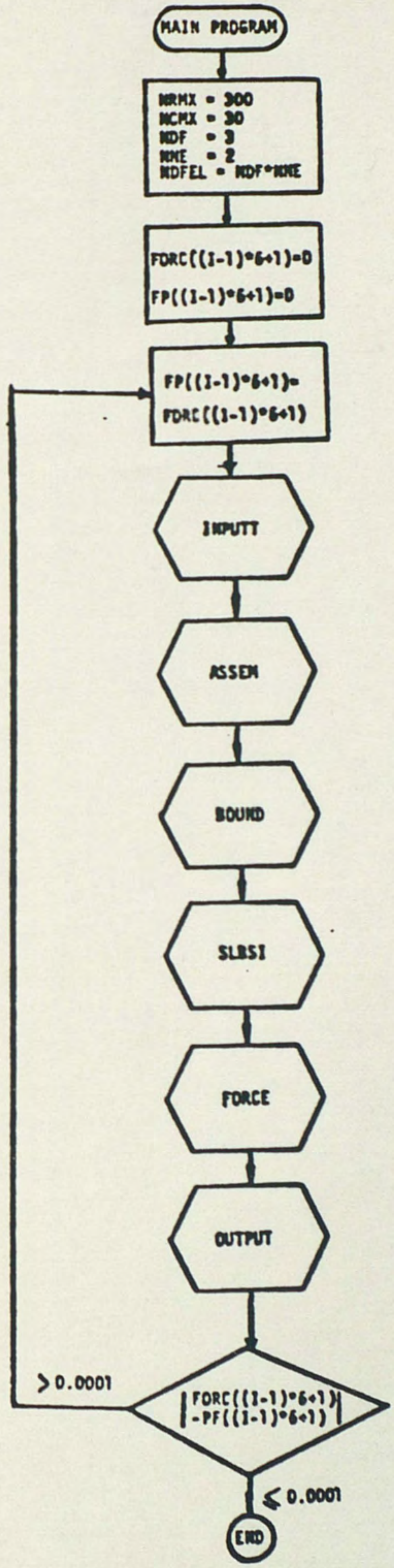


Figure 5. Flow chart diagram of main program.

The fundamental scheme of the program, which leads to the problem's solution, is as follows:

1 - Structural identification

2 - Computation of the element stiffness matrix

3 - Assembling of total stiffness matrix

4 - Applying boundary condition

5 - Solution of the system of equations

6 - Evaluation of the elemental forces

The input data, consisting of five sets of data cards, is formatted in the subroutine INPUTT which is sequenced in the following manner:

1 - Material properties of the structure

2 - Nodal locations with respect to global coordinate system

3 - Element properties and connectivity table

4 - Description of the applied load

5 - Boundary or support conditions

The total element stiffness matrix is evaluated in subroutine ASSEM and a brief flow chart of this subroutine is shown in Figure 6. This subroutine first calls the subroutine STIFF to compute the individual element stiffness matrix, and then by calling subroutine ELASS it will assemble the total stiffness matrix. However, it should be pointed out that the stiffness values used in each cycle have been modified using subroutine PDELTA. Calculating adjusting coefficients, shown in Table 1, the PDELTA subroutine accounts for the stability of the structure. At the beginning of subroutine STIFF the element

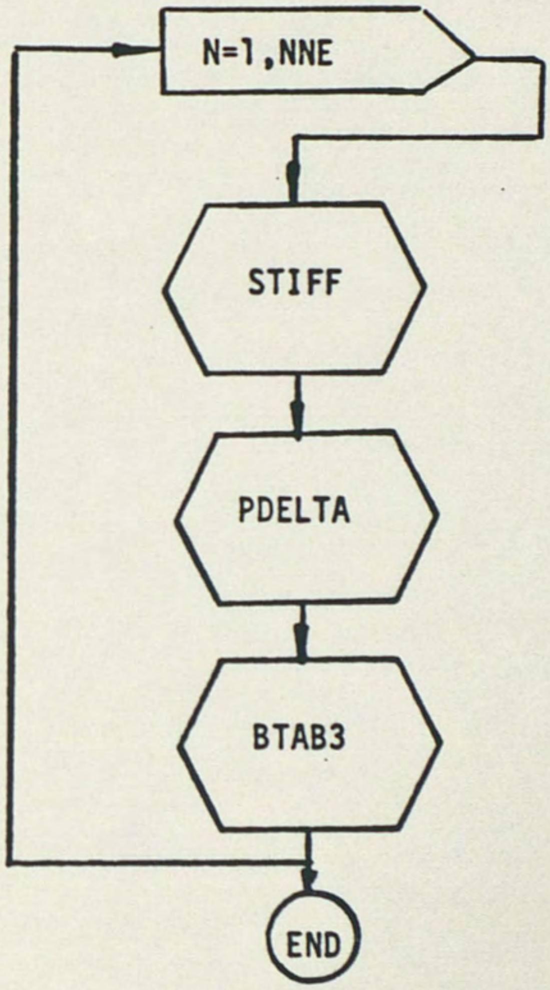


Figure 6. Flow chart of subroutine ASSEM.

rotation matrix is computed and placed in Array ROT. Then the terms of the local member stiffness matrix are computed according to Equation (5) and stored in Array ELST. Finally, this matrix is transferred to a global coordinate system. For this computation, subroutine ETAB3 is used. Notice that the global member stiffness matrix is also stored in Array ELST.

Once all the element matrices have been assembled into the global stiffness matrix, we introduce the boundary conditions and store them in subroutine BOUND.

The element nodal forces will be computed in subroutine FORCE. The computation of the member end forces is done within a loop from one to the total number of elements. These forces are stored in Array F and assembled into total member end forces in Array FORC. These member end forces refer to the local system, and then they are rotated to the global coordinated system, which are then stored in Array FG. Finally, these member contributions to the nodal resultants are added to the Array REAC.

The uniformly loaded members are integrated into fifty subelements and apply equivalent concentrated load at the center of each element and then, fixed end moments and shears for each element are calculated in subroutine FEM and stored in Array SUM and SUE, respectively. After the first cycle, their values are adjusted based on coefficients shown in Table 2.

It should be pointed out that the solution will be the nodal displacement at the node points. Final solution is a combination of solutions in each element lumped together at the common boundaries.

There are many methods of solving simultaneous equations, but the method adopted in this program is by the GAUSS elimination method, taking advantage of symmetric banded matrices. This is shown in subroutine SLBSI.

Finally, the subroutine OUTPT prints the nodal displacements and forces in X and Y coordinates the moment about the Z axis in both local and global coordinate systems.

Example 1

As an illustration of the application of second-order analysis, let us consider the case of the frame shown in Figure 7 which indicates the overall dimensions of all members.[8]

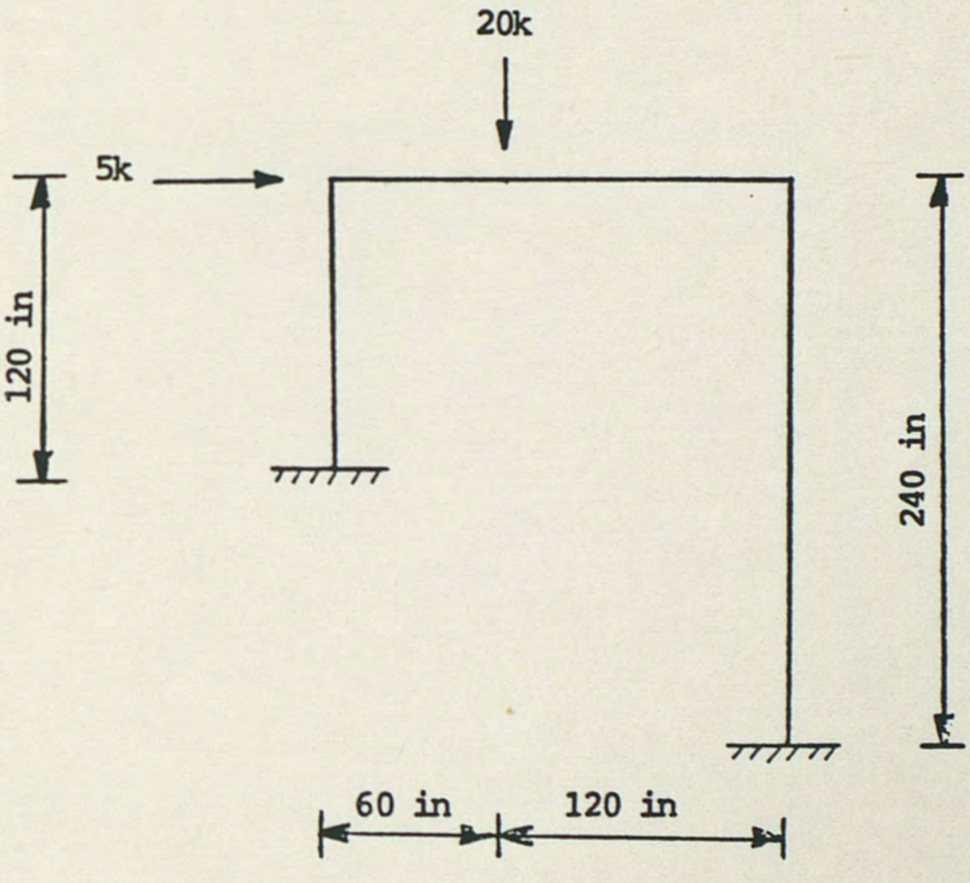


Figure 7. Frame geometry and loading condition.

For all members:

Cross sectional area = $3.174\ in^2$

Modulus of elasticity = 29000 Ksi

Moment of inertia = $7.233\ in^4$

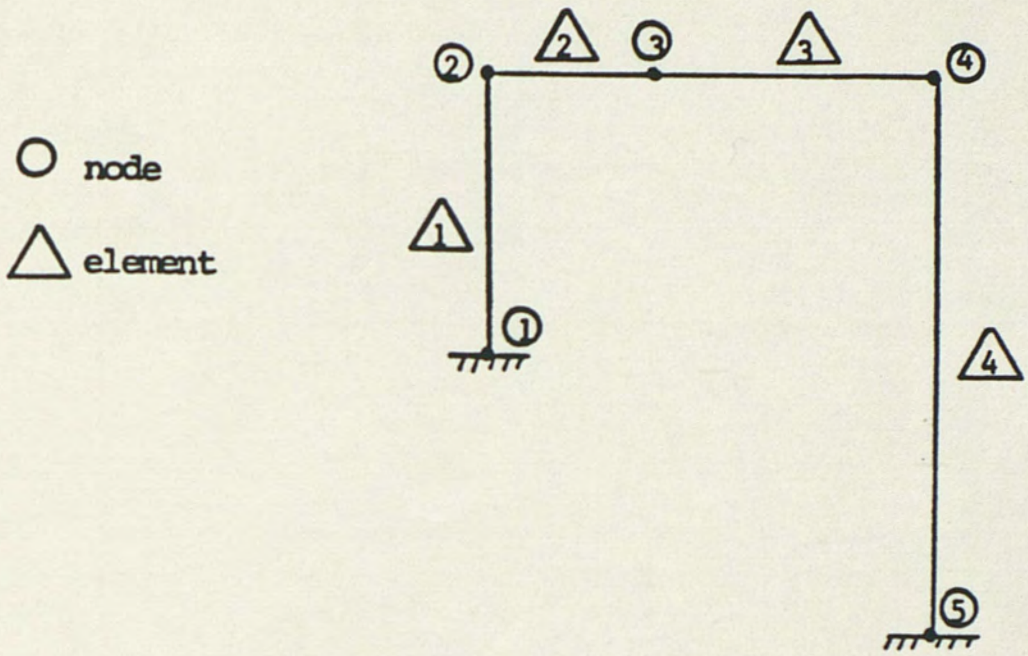


Figure 8. Finite element mesh used for frame shown in Figure 7.

Figure 8 indicates the node and element numbering used in the Finite Element Analysis. The results of first order and two cycles of a second order analysis are listed in Table 3.

The input and output results of this example are shown under Appendix III as a sample model.

TABLE 3

SECOND ORDER ANALYSIS OF FRAME

Moment	First-Order (K-in)	SECOND-ORDER 1st Cycle (K-in)	SECOND-ORDER 2nd Cycle (K-in)
M_1	357.91	438.42	437.67
M_2	-67.53	-11.68	-12.87
M_3	639.11	682.92	682.10
M_4	-347.58	-377.34	-375.85
M_5	271.65	311.42	310.72

The results of the 3rd cycle of second-order analysis are compared with the reference (8) and tabulated in Table 4.

TABLE 4

COMPARISON OF SECOND-ORDER ANALYSIS
RESULTS WITH REFERENCE (8)

Moment	Results of 3rd Cycle	Results of 3rd Cycle From Reference (8)	% Difference
M_1	437.63	436.81	0.187
M_2	-12.79	-10.50	17.904
M_3	681.98	681.79	0.027
M_4	-375.80	-371.41	1.168
M_5	310.66	302.82	2.523

The free body diagram for each member relating to the results of second order analysis is given in Figure 9.

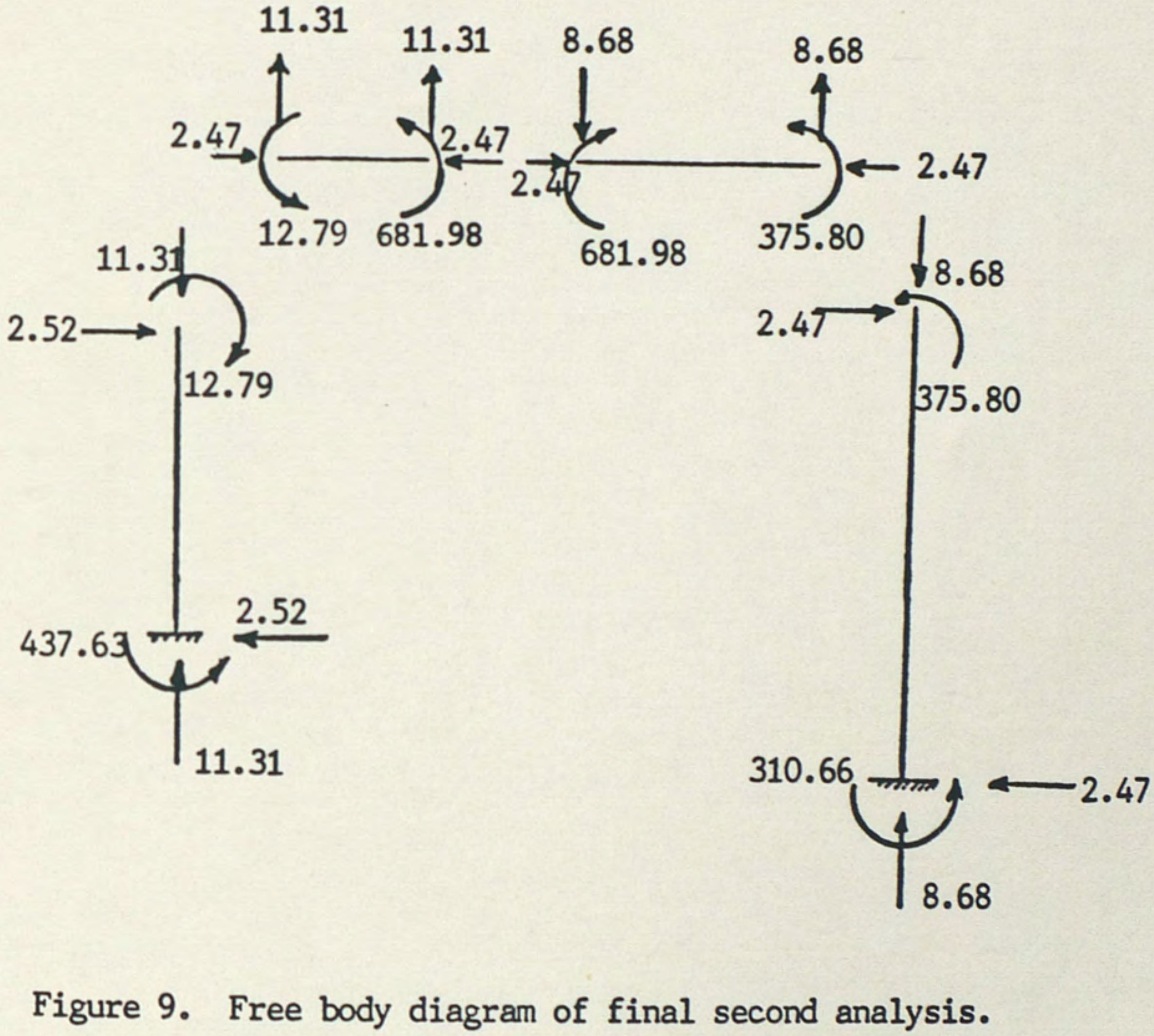


Figure 9. Free body diagram of final second analysis.

Final moment diagrams with and without inclusion of axial load effects is shown in Figure 10.

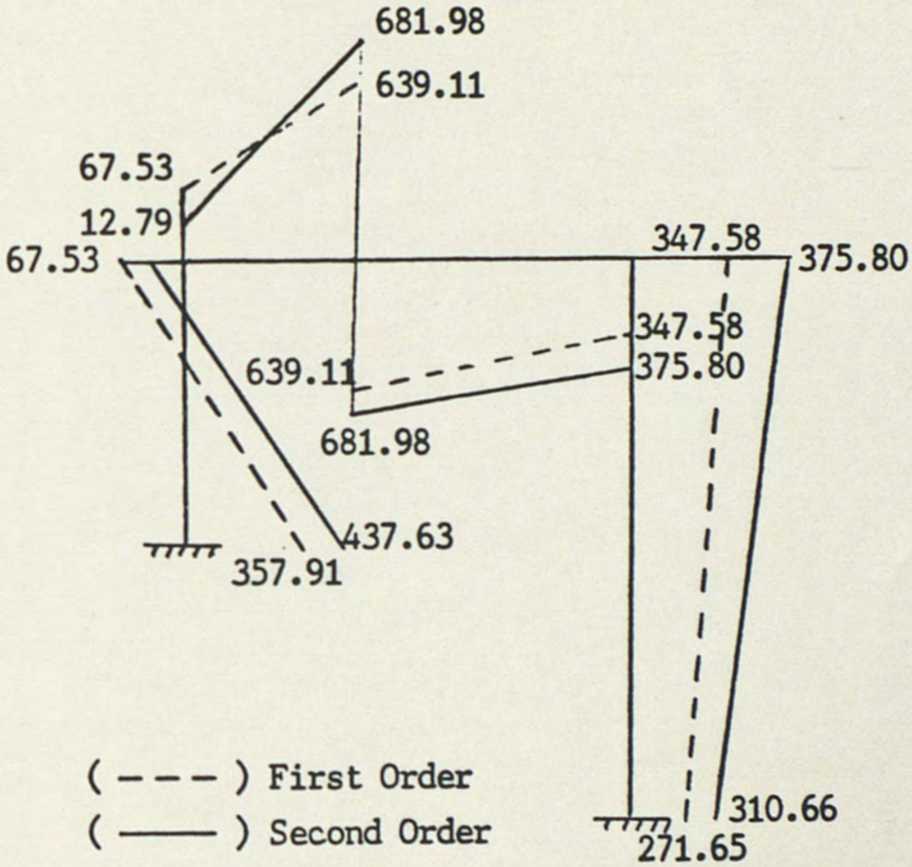
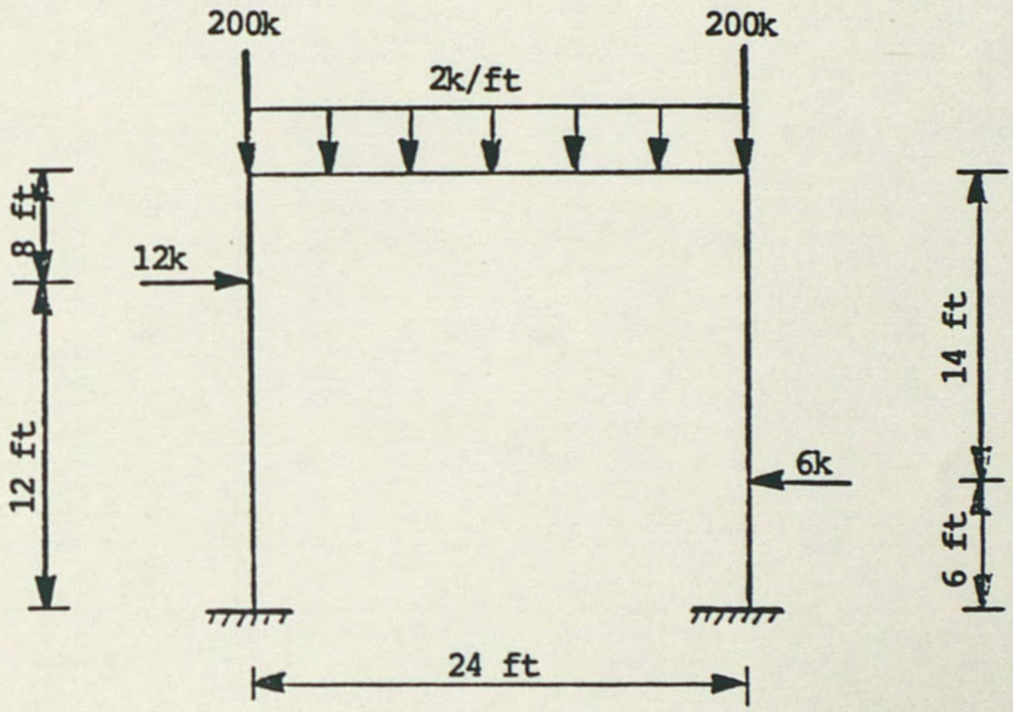


Figure 10. Final moment diagram.

Example 2

This problem is chosen from Reference (6) to compare the results for the frame shown in Figure 11, assuming that the effect of axial load upon its flexure is neglected.



$I = 250\ in^4$

$E = 30000$ Ksi

Figure 11. Frame geometry and loading condition.

The axial forces resulted from each cycle are used to modify the fixed-end moments and shears as well as stiffnesses for the next cycle.

Figure 12 indicates the node and element numbering used in Finite Element Analysis.

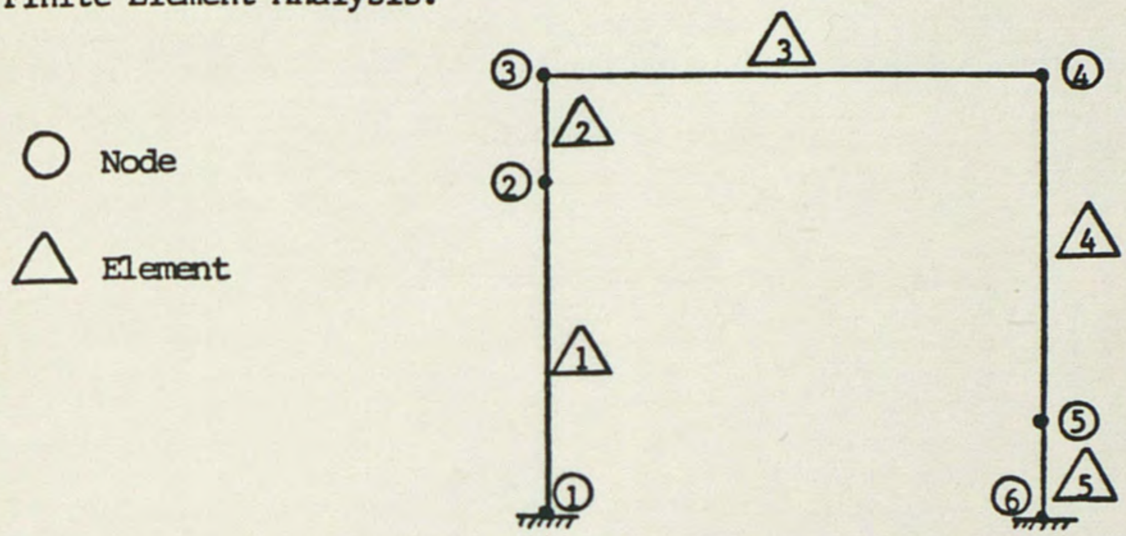


Figure 12. Finite element mesh used for frame shown in Figure 11.

The free body diagram for each member relating to the results of the second-order analysis is given in Figure 13.

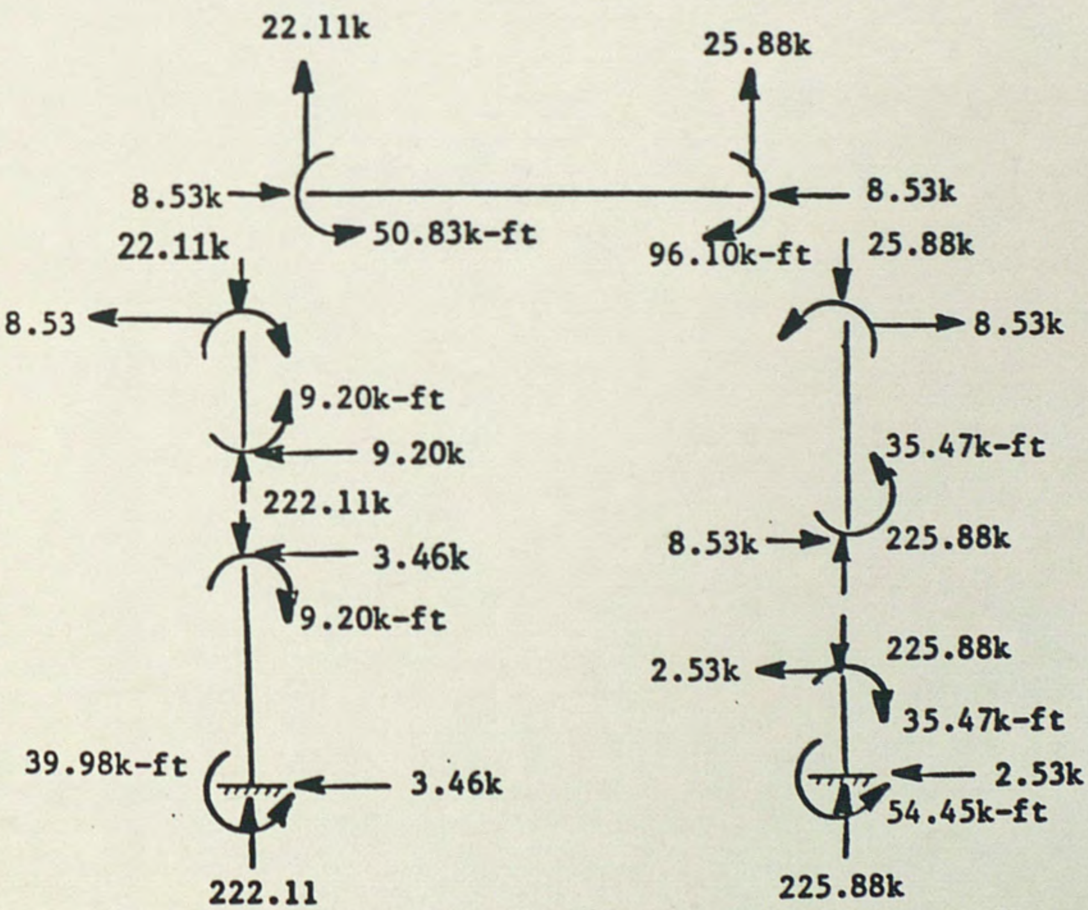


Figure 13. Final second-order results.

The results of the second-order analysis using the finite element computer program listed in Appendix II versus Reference (6) are shown in Table 5.

TABLE 5

COMPARISON OF SECOND-ORDER ANALYSIS RESULTS
FROM THIS PAPER WITH REFERENCE (6)

Moment	Results From this Paper K-ft	Results From Ref. (6) K-ft	% Difference
M_1	39.98	40.02	0.100
M_2	9.20	9.14	0.656
M_3	50.83	50.86	0.059
M_4	96.10	96.11	0.010
M_5	35.47	35.44	0.084
M_6	54.45	54.45	0.000

Figure 14 shows the final moment diagram with and without the effect of axial force.

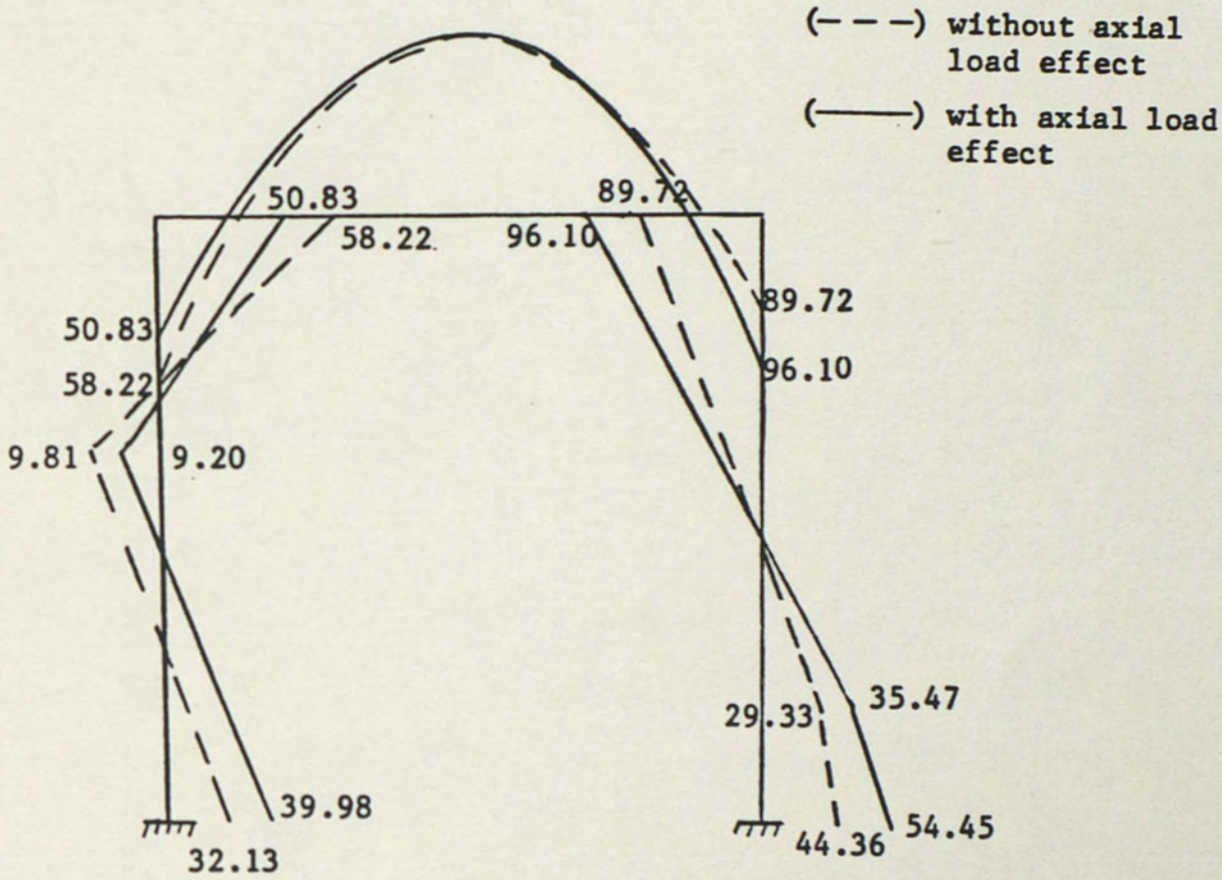


Figure 14. Final moment diagram.

Example 3

An illustration of the effect of secondary moments on the frame is shown in Figure 15 with three different axial forces.

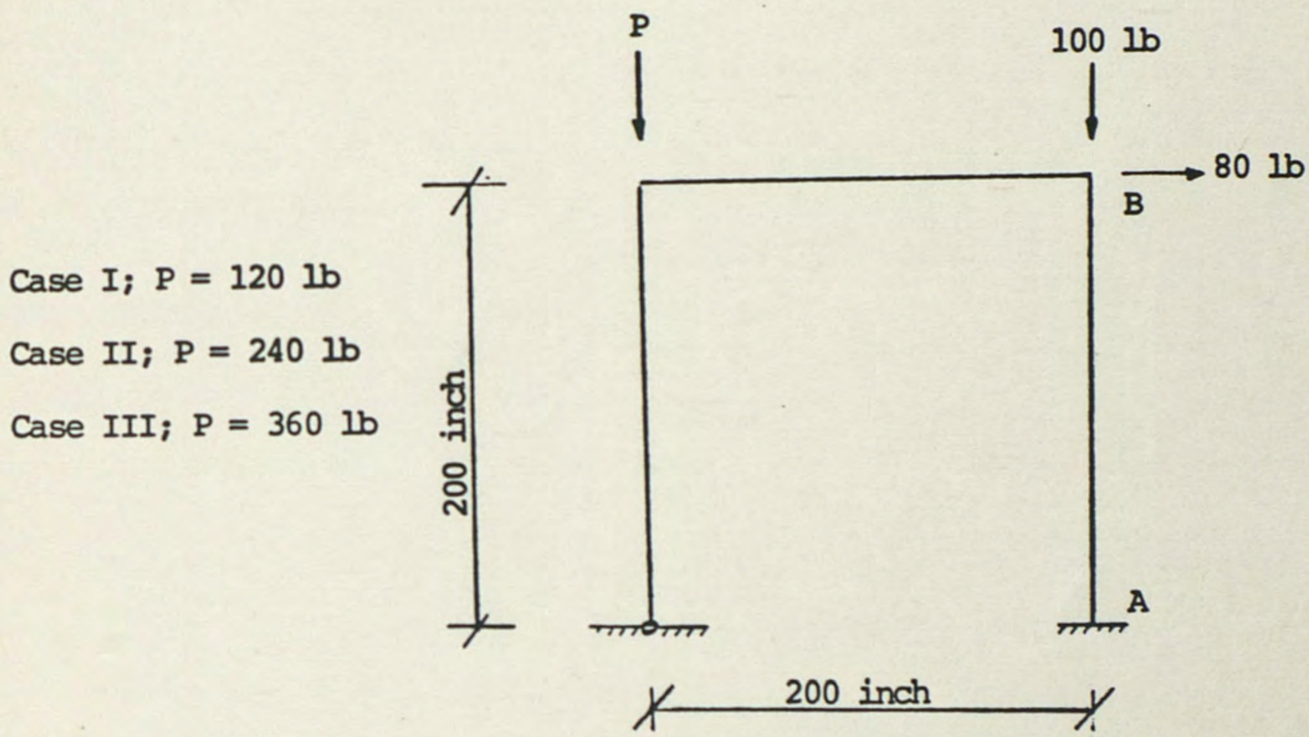


Figure 15. Frame geometry and loading condition used in examples.

For all members:

Cross sectional area = 30 inch^2

Modulus of elasticity = 29000 Ksi

Moment of inertia = 2.25 in^4

Figure 16 indicates the node and element numbering used in Finite Element Analysis for all three cases.

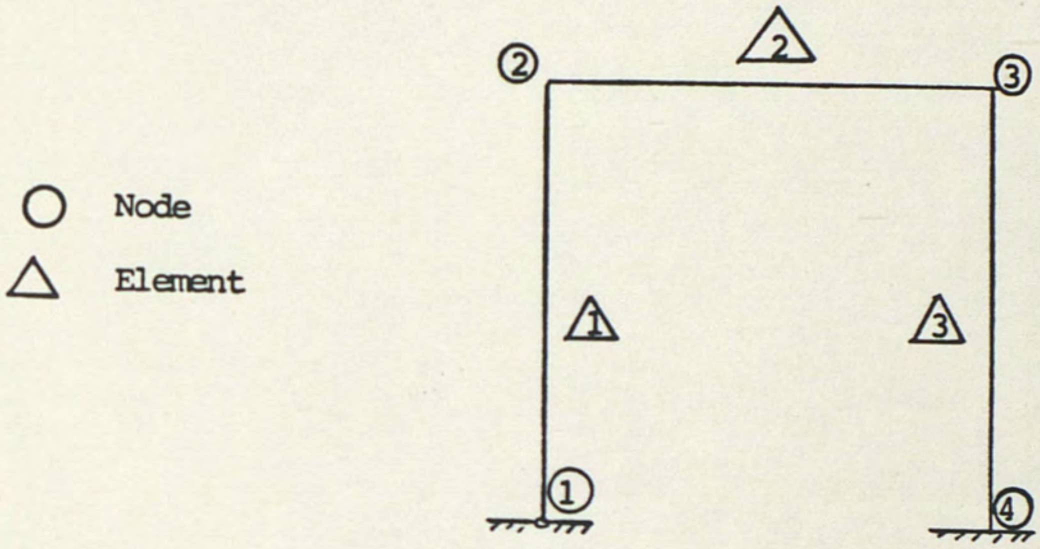


Figure 16. Finite element mesh used for frame shown in Figure 15.

The final free body diagram of each member relating to the results of second-order analysis for Case I, Case II and Case III are shown in Figures 17, 18, and 19, respectively.

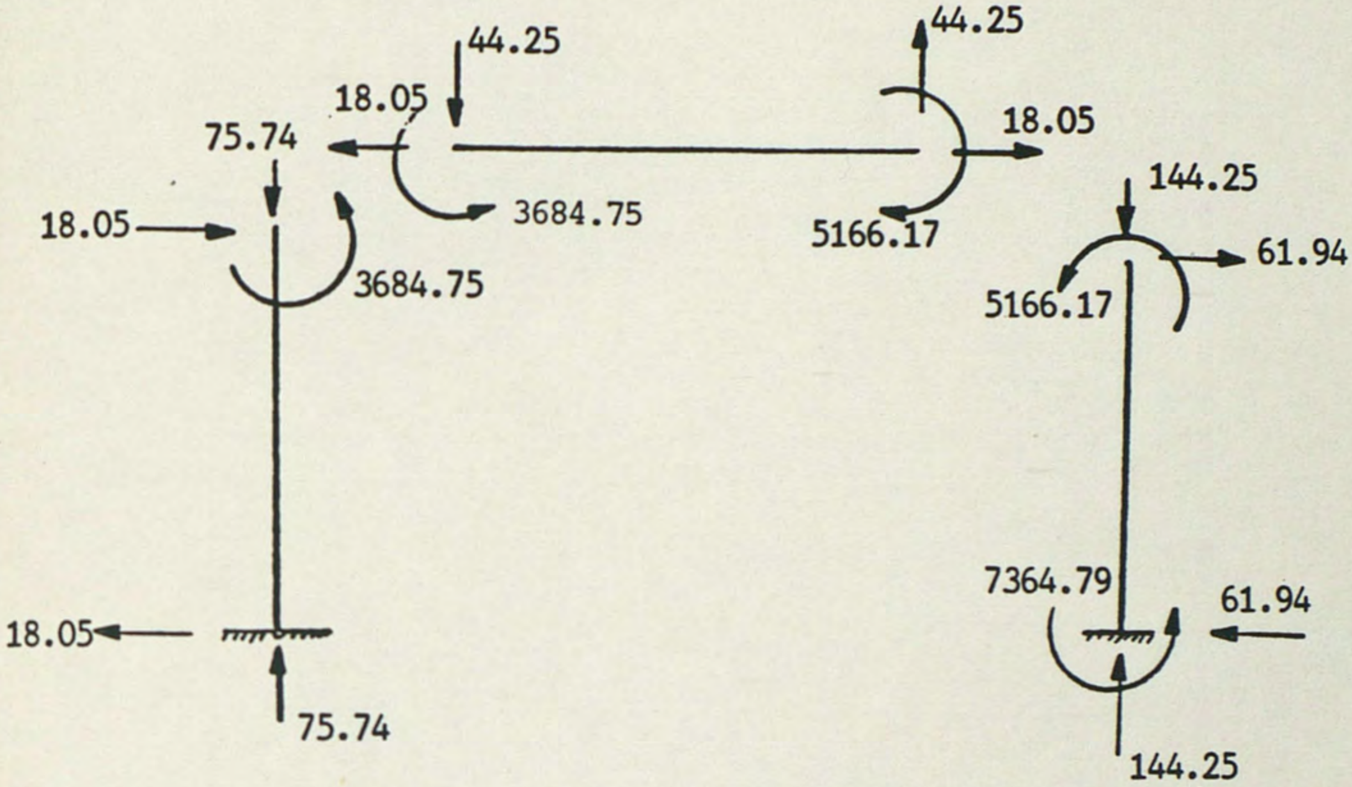


Figure 17. Free body diagram of final second order analysis for Case I (P = 120 lb).

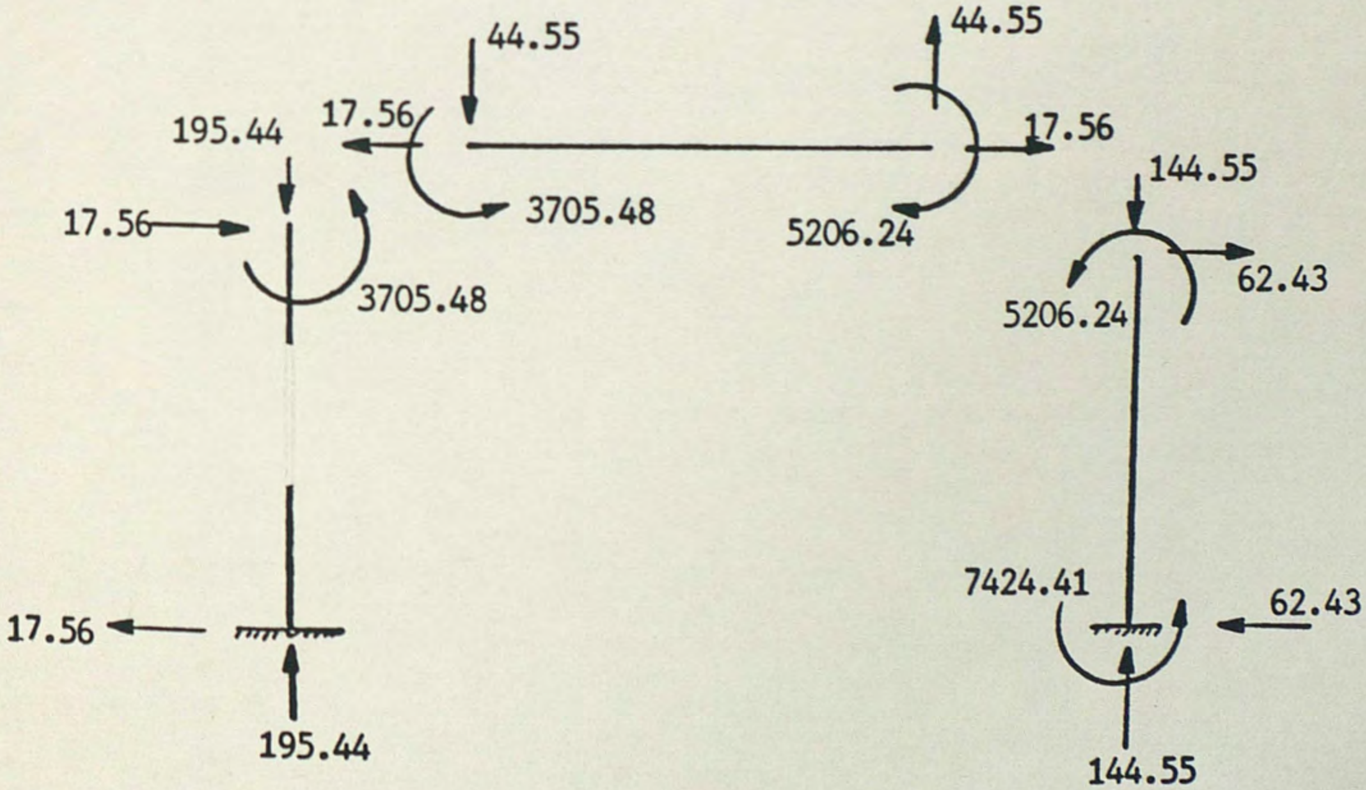


Figure 18. Free body diagram of final second order analysis for Case II (P = 240 lb).

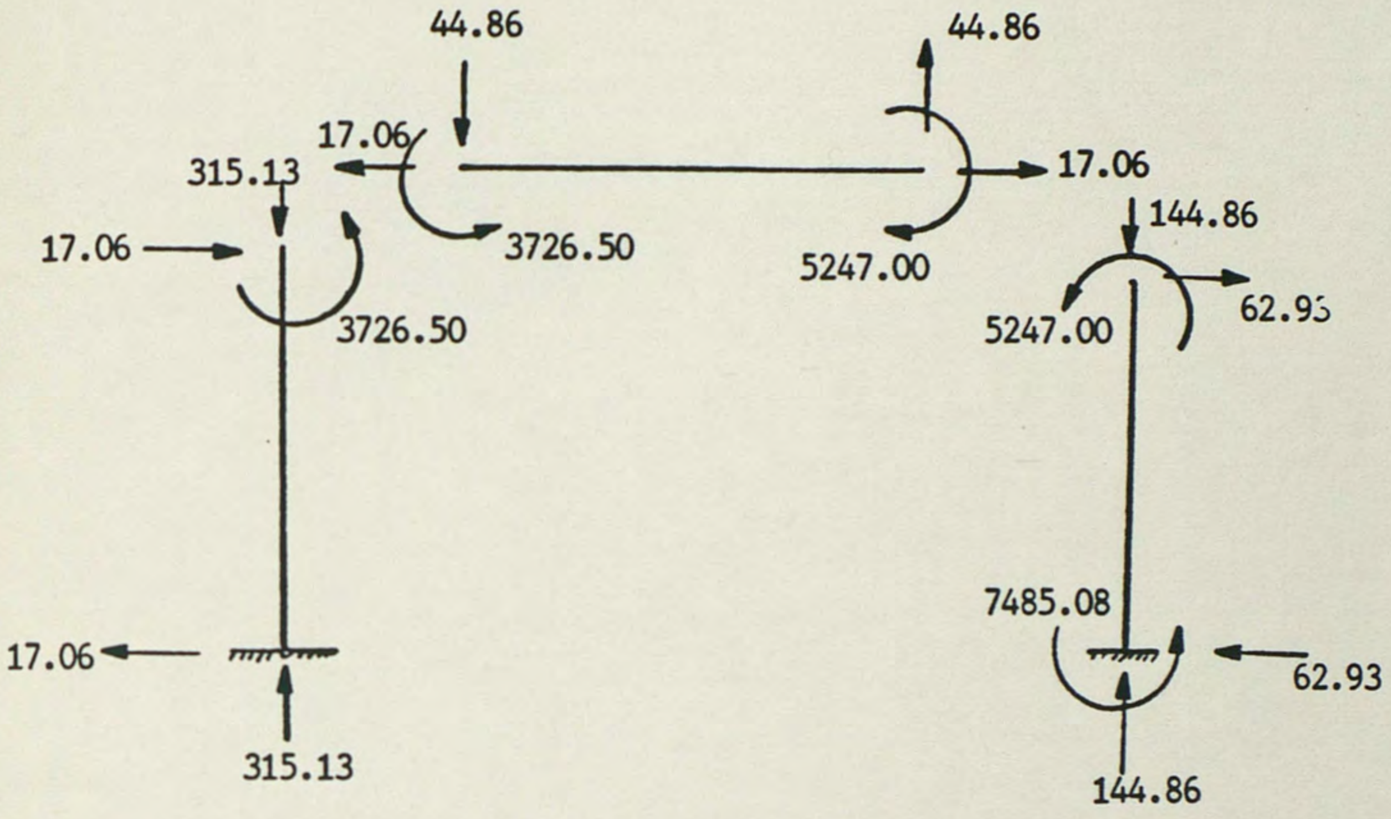


Figure 19. Free body diagram of final second order analysis for Case III (P = 360 lb).

The results of first order and second order analysis for all three cases are summarized in Table 6.

TABLE 6

RESULTS OF FIRST AND SECOND ORDER ANALYSIS FOR ALL THREE LOAD CASES

Moment (lb-inch)	First Order Analysis	Second Order Analysis		
	For All Three Cases	Case I P=120 lb	Case II P=240 lb	Case III P=360 lb
M_1	0.00	0.00	0.00	0.00
M_2	3636.37	3684.75	3705.48	3726.50
M_3	5090.92	5166.17	5206.24	5247.00
M_4	7272.70	7364.79	7424.41	7485.08

As the results of second-order analysis indicate, the moments in column AB for the three cases are different, which can yield to three different column selections. However, the results of first-order analysis show the same moments for their load cases, and using the present K method, which was presented in an earlier section of this report, the value of K, as obtained from the alignment chart, will be the same for all cases:

G_A = 1.0 for fixed-end condition

$$G_B = \frac{I_c/L_c}{I_g/L_g} = \frac{2.25/200}{2.25/200} = 1.0$$

Therefore, K = 1.3 which will lead to the same column selection for column AB in all three load cases.

It is clear then if the column is adequate for Case II, it must be overdesigned for Case I and underdesigned for Case III. Thus by using the second-order analysis, a more efficient design can be obtained.

CHAPTER V

CONCLUSIONS AND COMMENTS

In general, neglecting the effect of axial force on deformed structure (first-order analysis) overestimates the stiffness and the overall strength of the structure. These effects are compensated for in the current allowable-stress technique by using the effective column length and C_m factors in the design of members. As it was demonstrated in an earlier section of this report, the effective length factor can lead to an over or underestimate of the member size.

A more reliable technique for calculating these secondary moments is introduced in this paper along with a finite element computer program. The derivation of adjusting coefficients for secondary effects is based on stability function and critical load of each member. The computer program will provide nodal displacements and element forces for first-and second-order analysis of the frame structure. Generally, the compressive axial load tends to increase the deformation and moment which is tantamount to a reduction in the strength of the member; on the other hand, tensile axial load will slightly increase the strength of the member.

The final solution obtained by cyclic fashion and the convergence is extremely fast and normally the third and fourth iteration produces acceptable results (if it does not converge within seven or eight trials, there is a strong possibility that the structure is unstable).

The example problems in this paper were performed with a 64 bit of accuracy. However, the author would caution a 32 bit computer can cause losing the accuracy of results because the coefficients used to modify the stiffness matrix were calculated in terms of sinh and cosh functions and have a very small magnitude. Thus, a double precision processor is recommended.

Also, by using a mini computer, the execution time can easily be noticeable due to cyclic techniques used to calculate the secondary moments.

At the present time, the computer program is capable of solving two dimensional frame problems with concentrated and uniform load conditions. However, it can easily be extended to account for members with trapezoid type loading. Also, if the axial force in any member equals the load where buckling occurs, a singular matrix occurs and the results cannot be obtained. This can be a base for further improvement of the program to calculate the buckling load of each member.

APPENDIX I

STABILITY FUNCTION

The coefficients involved in the effects of secondary moments on the stiffness and fixed end action shown in Tables 1 and 2 were developed based on stability function and critical load of each member.[6]

The development of some of these coefficients are introduced in the following three parts which include A) Rotational functions; B) Translational function; and C) Fixed end action.

Rotational Functions

In order to develop the stiffness coefficients, consider prismatic or uniform beam element i which has no deformation when it is unloaded with length L_i.

At the joint 1, moment K_{11} and at the joint 2, restraining moment K_{21} act. The shear forces at end of the beam shown in Figure 20 would be:

$$K_{12} = K_{22} = - \frac{K_{11} + K_{21}}{L_i} \tag{7}$$

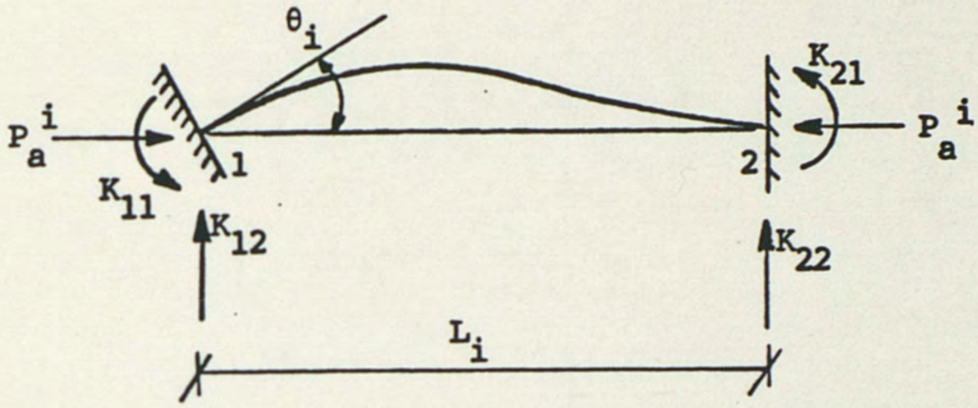


Figure 20. Beam element with axial force P_i.

By applying the analysis of buckling load based on the differential equation of an elastic curve, we can write the following equation for the free body diagram shown in Figure 21:

$$EI \frac{d^2y}{dx^2} = M = -(P_a^{\,i} Y + K_{11} - K_{12} X) \tag{8}$$

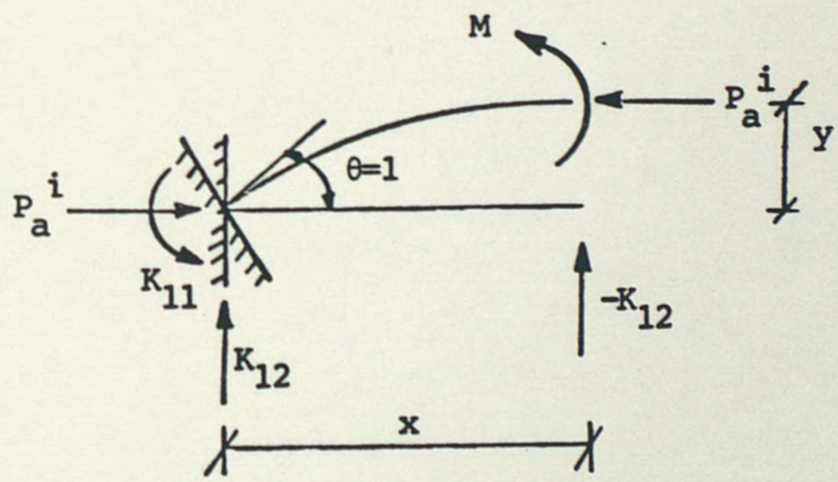


Figure 21. Free body diagram of beam element i.

where P_a^i is the internal force applied to element i. By replacing the value of K_{12} from Equation (7), we can rewrite Equation (8) as follows:

$$EI \frac{d^2y}{dx^2} = -P_a^i Y - K_{11} + (K_{11} + K_{21}) \frac{X}{L_i} \qquad (9)$$

The lateral load P_a^i can be expressed as a scalar multiple of P_E^i, the pinned end Euler load for beam element i for the buckling in the plane of the applied unit rotation.

$$P_a^i = \lambda_i \, P_E^i \qquad (10)$$

Since the critical or buckling load for a pinned end uniform member is $P_E^i = \frac{\pi^2 EI}{L_i^2}$, therefore, Equation (10) becomes:

$$P_a^i = \lambda_i \, \pi^2 (\frac{EI}{L_i^2}) \qquad (11)$$

Substituting Equation (11) into Equation (9) and rearranging gives:

$$\frac{d^2y}{dx^2} + \frac{\pi^2 \lambda i}{L_i^2} Y = \frac{(K_{11} + K_{21})(\frac{x}{L_i}) - K_{11}}{EI} \qquad (12)$$

The general solution of Equation (12) can be expressed as:

$$Y = A \operatorname{Sin}(\frac{KL_i}{L_i} X) + B \operatorname{Cos}(\frac{KL_i}{L_i} X) + \frac{L_i^2}{(KL_i)^2 \, EI} [(K_{11} + K_{21})$$

$$\frac{x}{L_i} - K_{11}] \qquad (13)$$

where: $$KL_i = \pi\sqrt{\lambda_i} \tag{14}$$

Substituting the boundary conditions $Y = 0$ at $X = 0$ and $Y = 0$ at $X = L_i$ yields values for the constants A and B:

$$A = -\frac{L_i}{(KL_i)^2\,EI}\,[K_{11}\,\mathrm{Cot}(KL_i) + K_{21}\,\mathrm{CSC}\,(KL_i)] \tag{15}$$

and

$$B = \frac{L_i^2}{(KL_i)^2\,EI}\,K_{11} \tag{16}$$

Substituting these values of A and B into Equation (13) yields:

$$\frac{(KL_i)^2\,EI}{L_i^{\,2}}\,Y = -\,[K_{11}\,\mathrm{Cot}(KL_i) + K_{21}\,\mathrm{CSC}\,(KL_i)]\,\mathrm{Sin}\,(\frac{KL_i}{L_i}\,X) + K_{11}\,\mathrm{Cos}(\frac{KL_i}{L_i}\,X) + (K_{11} + K_{21})\,\frac{X}{L_i} - K_{11} \tag{17}$$

Differentiating Equation (17) with respect to X in order to obtain the slope of the elastic curve:

$$\frac{(KL_i)^2\,EI}{L_i}\,\frac{dy}{dx} = K_{11}[1 - (KL_i)\,\mathrm{Sin}\,(\frac{KL_i}{L_i}\,X) - (KL_i)\,\mathrm{Cot}\,(KL_i)\,\mathrm{Cos}(\frac{KL_i}{L_i}\,X)] + K_{21}[1 - (KL_i)\,\mathrm{CSC}\,(KL_i)\,\mathrm{Cos}\,(\frac{KL_i}{L_i}\,X)] \tag{18}$$

Substituting the boundary condition $\frac{dy}{dx} = 0$ at $X = L_i$ into Equation (18), the relation between end moments would be:

$$K_{21} = \frac{(KL_i) - \text{Sin}\,(KL_i)}{\text{Sin}\,(KL_i) - (KL_i)\,\text{Cos}\,(KL_i)} K_{11} \tag{19}$$

Now by considering $\frac{dy}{dx} = 1$ at $X = 0$ along with Equation (19) into Equation (18), the value of the restraining moment at left end of Figure 2 is given by:

$$K_{11} = \frac{(KL_i)[\text{Sin}(KL_i) - (KL_i)\,\text{Cos}(KL_i)]}{2(1 - \text{Cos}(KL_i)) - (KL_i)\,\text{Sin}(KL_i)} \left(\frac{EI}{L_i}\right) \tag{20}$$

Now letting

$$fc = 2(1 - \text{Cos}(KL_i)) - (KL_i)\,\text{Sin}(KL_i) \tag{21}$$

Equation (20) can be rewritten as:

$$K_{11} = \frac{KL_i\,(\text{Sin}\,KL_i - KL_i\,\text{Cos}\,KL_i)}{fc} \frac{EI}{L_i} \tag{22}$$

By substituting Equation (22) into (19)

$$K_{21} = \frac{(KL_i) - \text{Sin}(KL_i)}{\text{Sin}(KL_i) - (KL_i)\,\text{Cos}(KL_i)} * \frac{KL_i(\text{Sin}\,KL_i - KL_i\,\text{Cos}\,KL_i)}{fc} \left(\frac{EI}{L_i}\right) \tag{23}$$

Thus:

$$K_{21} = \frac{KL_i(KL_i - \text{Sin}\,KL_i)}{fc} \frac{EI}{L_i} \tag{24}$$

For zero axial force, KL_i becomes zero, by applying La'Hospital's rule, which states that in attempting to obtain the limit of the quotient

$$\frac{f(KL_i)}{g(KL_i)}$$

Where both functions approach zero as KL_i approaches zero, one may instead consider the quotient

$$\frac{f'(KL_i)}{g'(KL_i)}$$

where $f'(KL_i)$ and $g'(KL_i)$ are first derivatives of $f(KL_i)$ and $g(KL_i)$ with respect to KL_i.

$$\left[\frac{KL_i(\text{Sin } KL_i - KL_i \text{ Cos } KL_i)}{2(1 - \text{Cos } KL_i) - KL_i \text{ Sin } KL_i}\right]_{\lambda = 0} = 4$$

and

$$\left[\frac{KL_i - \text{Sin }(KL_i)}{\text{Sin } KL_i - KL_i \text{ Cos } KL_i}\right]_{\lambda = 0} = 1/2$$

By using the above set of formulas, Equations (22) and (24) can reduce to:

$$K_{11} = \frac{4\ EI}{L_i} \tag{25}$$

$$K_{21} = \frac{2\ EI}{L_i} \tag{26}$$

Thus, it is seen for the case of zero axial force, Equations (24) and (25) represent the first-order analysis bending stiffness for a prismatic member. The shear forces at the end of the member can also be represented by substituting the value of K_{11} and K_{21} into Equation (7) which leads to:

$$K_{12} = -K_{22} = \frac{K_{11} + K_{21}}{L_i}$$

or

$$K_{12} = -K_{22} = \frac{EI}{L_i} \left[\frac{KL_i(\text{Sin } KL_i - KL_i \text{ Cos } KL_i)}{fc} + \frac{KL_i(KL_i - \text{Sin } KL_i)}{fc}\right] \tag{27}$$

$$K_{12} = -K_{22} = \frac{EI}{L_i} \left[\frac{(KL_i)^2(1 - \text{Cos } KL_i)}{fc}\right] \tag{28}$$

A unit rotation at node right and of Figure 2 could lead to the same stiffness values of the left end.

For tensile axial forces, λ_i would have negative value and variable KL_i can be defined as:

$$KL_i = \pi\sqrt{\lambda_i} = i\pi \sqrt{|\lambda_i|} \tag{29}$$

where i represents the imaginary unit. Since KL_i is a complex variable, the following functions are defined:

$$\text{Sin}(KL_i) = \frac{e^{i(KL_i)} - e^{-i(KL_i)}}{2i} \tag{30a}$$

and

$$\text{Cos}(KL_i) = \frac{e^{i(KL_i)} + e^{-i(KL_i)}}{2} \tag{30b}$$

$$\text{Sinh}(KL_i) = \frac{e^{KL_i} + e^{KL_i}}{2} \tag{30c}$$

$$\text{Cosh}(KL_i) = \frac{e^{KL_i} + e^{-KL_i}}{2} \tag{30d}$$

By using the above relations into Equations (22), (24), and (28) and after simplification, the following relationships could be written for stiffness matrix with tensile force:

$$K_{11} = \frac{KL_i[\text{Sinh }(KL_i) - KL_i]}{2\text{-}2 \text{ Cosh } KL_i + KL_i \text{ Sinh } KL_i} \; \frac{EI}{L_i} \tag{31}$$

$$K_{21} = \frac{KL_i(KL_i \text{ Cosh } KL_i - \text{Sinh } KL_i)}{2\text{-}2 \text{ Cosh } KL_i + KL_i \text{ Sinh } KL_i} \frac{EI}{L_i} \tag{32}$$

$$K_{12} = \frac{(KL_i)^2 \; (\text{Cosh } KL_i - 1)}{2\text{-}2 \text{ Cosh } KL_i + KL_i \text{ Sinh } KL_i} \frac{EI}{L_i} \tag{33}$$

Translation Function

The rotation functions are related with a node rotation. Another operation is a node translation, Figure 22. Node 1 is translated through a unit distance relative to Node 2. The translation operation may be alternatively regarded as the rotation of the 1 and 2 through angles of $\frac{1}{L_i}$ during which the restraining moments K_{11} and K_{21} remain unchanged. The lateral force in Figure 23 differs somewhat from the lateral force in Figure 22. But for a small angle of translation the difference may be neglected.

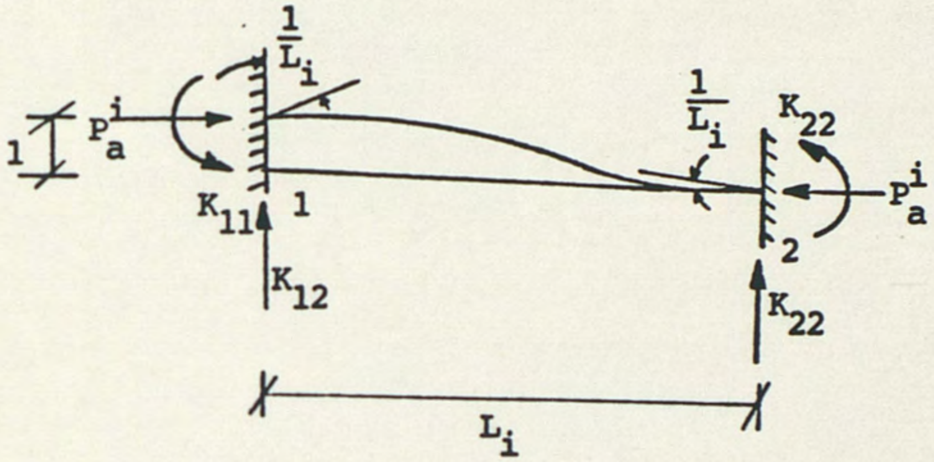


Figure 22. Free body diagram, beam element i with unit translation of end 1.

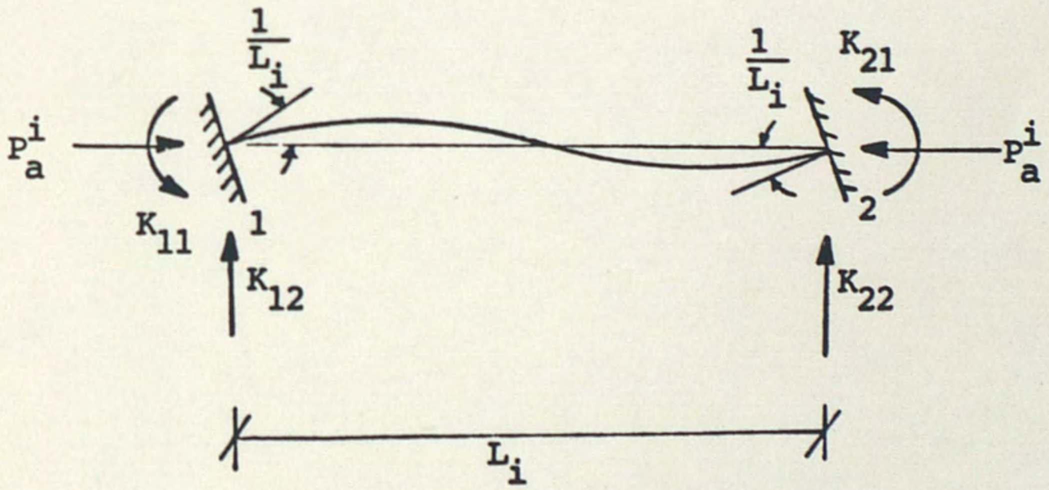


Figure 23. Rigid body rotation of beam element i.

Taking moments about Node 2 in Figure 22:

$$K_{12} = \frac{K_{11} + K_{21} - P_a^{\,i}(1)}{L_i} \tag{34a}$$

From equilibrium condition it is also clear to show:

$$K_{21} = -K_{22}$$

or

$$K_{21} = \frac{+P_a^{\,i}(1) - (K_{11} + K_{21})}{L_i} \tag{34b}$$

The rotational relation for members shown in Figure 23 are:

$$K_{11} = K_{21} = \left[\frac{(KL_i)^2(1-\text{Cos } KL_i)}{fc}\right]\frac{EI}{L_i} \tag{35}$$

Replacing Equation (35) into (34a):

$$K_{12} = -K_{22} = 2\left[\frac{(KL_i)^2(1-\text{Cos } KL_i)}{fc}\right.\frac{EI}{L_i^{\,2}} - \frac{P_a^{\,i}}{L_i} \tag{36}$$

By following the procedure shown in rotational function, it could be shown:

for compressive axial force:

$$K_{12} = -K_{22} = \left[\frac{(KL_i)^3 \text{ Sin } KL_i}{12(2-2 \text{ Cos } KL_i - KL_i\text{Sin } KL_i)}\right]\frac{EI}{L_i^{\,3}} \tag{37}$$

and for tensile axial force:

$$K_{12} = -K_{22} = [\frac{(KL_i)^3 \text{ Sinh } KL_i}{2-2 \text{ Cosh } KL_i + KL_i \text{Sinh } KL_i}] \frac{EI}{L_i{}^3} \qquad (38)$$

The member stiffness matrix as presented by Equation (5) was defined by rotational and translational function which was shown in previous sections. Axial deformation of the member has been neglected.

Fixed End Actions

Let's look at a concentrated load shown in Figure 24. Member i is subjected to compressive axial force $P_a{}^i$. The ends ϕ and 2 are fixed against rotation and translation with moments K_{11} and K_{21} and shear forces K_{12} and K_{22}.

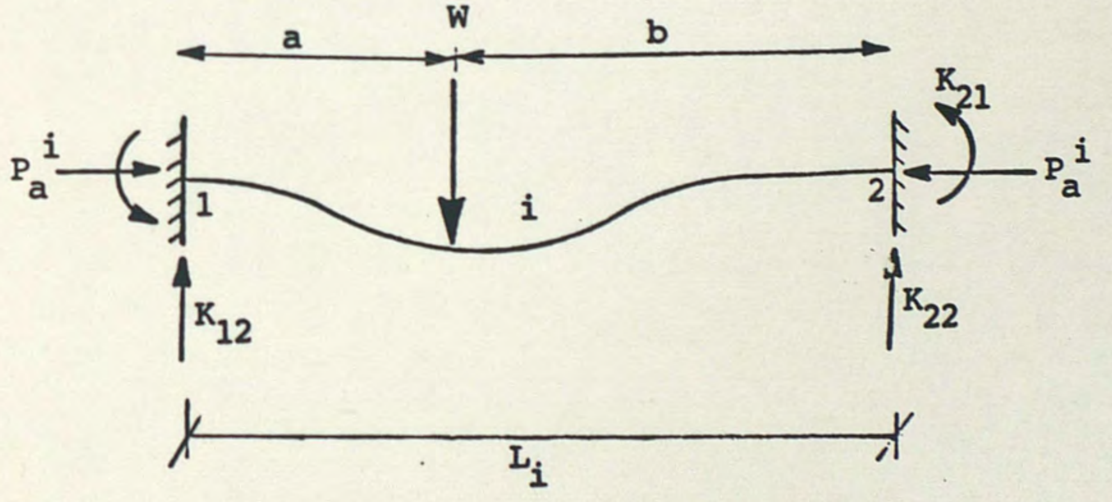


Figure 24. Prismatic beam element i with concentrated load W.

Applying the Equation of elasticity, the free body diagram is shown in Figure 25.

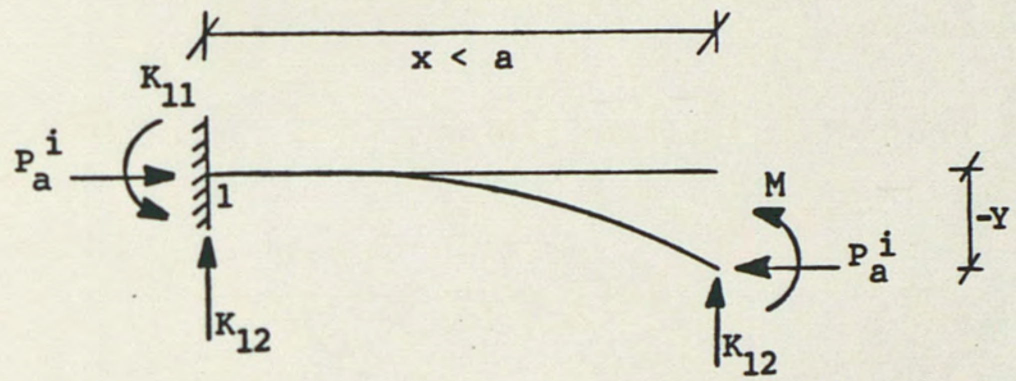


Figure 25. Free body diagram, x distance from end 1 when x < a.

$$EI \frac{d^2y}{dx^2} = M = - [K_{11} - P_a^{\,i}(-Y) - K_{12}(X)] \tag{39}$$

The general solution of Equation (39) would be:

$$Y = A \operatorname{Sin} (\frac{KL_i}{L_i} X) + B \operatorname{Cos} (\frac{KL_i}{L_i} X) - \frac{L_i^{\,2}}{(KL_i)^2 \, EI} [K_{11} - K_{12} (X)] \tag{40}$$

Take a derivative respect to X to yield:

$$\frac{dy}{dx} = \frac{KL_i}{L_i} A \operatorname{Cos}(\frac{KL_i}{L_i} X) - \frac{KL_i}{L_i} B \operatorname{Sin} (\frac{KL_i}{L_i} X) + \frac{L_i^{\,2}}{(KL_i)^2 \, EI} K_{12} \tag{41}$$

By considering the boundary condition:

$$\text{At } X = 0$$

$$Y = 0$$

$$\frac{dy}{dx} = 0$$

Therefore:

$$A = \frac{L_i^{\,3}}{(KL_i)^3 \, EI} K_{12} \tag{42a}$$

and

$$B = \frac{L_i^{\,2}}{(KL_i)^2 \, EI} K_{11} \tag{42b}$$

By substituting the values of A and B into Equation (40) and (41) and rearranging, the following expression can be obtained:

$$\frac{(KL_i)^2}{L_i^{\,2}} EIY = K_{11} \text{ Cos } (\frac{KL_i}{L_i} X) - \frac{L_i(K_{12})}{KL_i} \text{ Sin } (\frac{KL_i}{L_i} X)$$

$$+ K_{12} (X) - K_{11} \qquad 0 \leq X \leq a \tag{43}$$

and

$$\frac{(KL_i)^2 \, EI}{L_i^{\,2}} \frac{dy}{dx} = - \frac{KL_i}{L_i} K_{11} \text{ Sin } (\frac{KL_i}{L_i} X) -$$

$$K_{12} \text{ Cos } (\frac{KL_i}{L_i} X) + K_{12} \qquad 0 \leq X \leq a \tag{44}$$

The free body diagram for the same member, when distance X from end 1 would be greater than a, is shown in Figure 26.

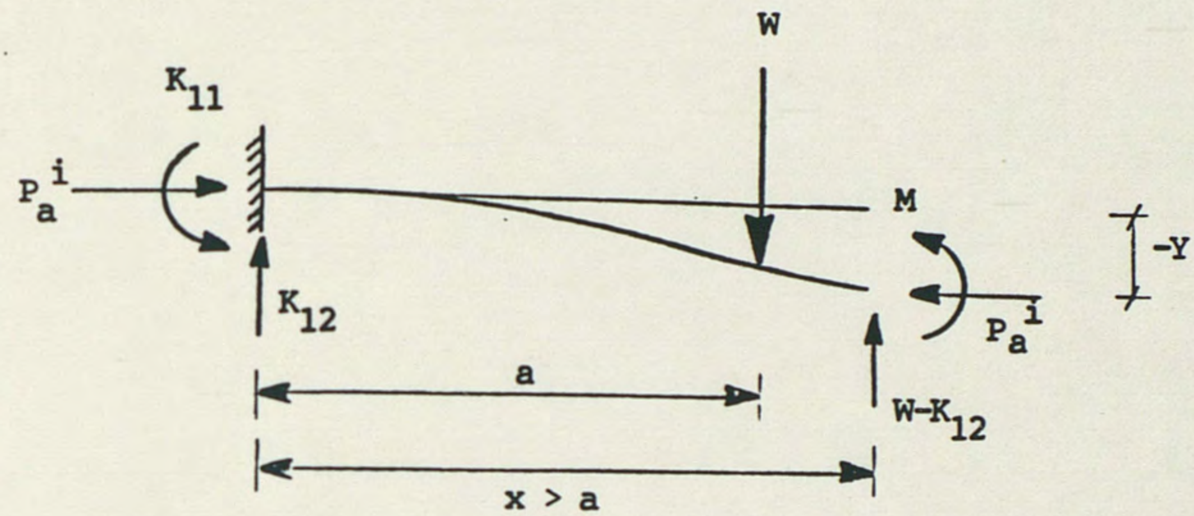


Figure 26. Free body diagram, x distance from end 1 when x > a

By applying the differential equation elasticity for the free body diagram shown in Figure 26 yields:

$$EI\,\frac{d^2y}{dx^2} = M = -[K_{12} - P_a^{\,i}(-Y) - K_{12}\,(x) + (-W)(x-a)] \qquad x \geq a \tag{45}$$

And the general solution of Equation (45) can be shown as:

$$Y = A\,\text{Sin}\,(\frac{KL_i}{L_i}X) + B\,\text{Cos}\,(\frac{KL_i}{L_i}X) - \frac{L_i^{\,2}}{(KL_i)^2\,EI}\,[K_{11} + W(a) - (K_{12} + W)X] \qquad x > a \tag{46}$$

Differentiating Equation (47) with respect to x yields:

$$\frac{dy}{dx} = \frac{KL_i}{L_i} A \text{ Cos } (\frac{KL_i}{L_i} x) - \frac{KL_i}{L_i} B \text{ Sin } (\frac{KL_i}{L_i} x) + \frac{{L_i}^2}{(KL_i)^2 \text{ EI}}$$

$$(K_{12} + W) \qquad (47)$$

Equating the impressions for $\frac{dy}{dx}$ from Equations (44) and (47) and the expression for Y from Equations (43) and (46) for the two segment lengths a and b at x = a yields:

$$A = \frac{-{L_i}^3}{(KL_i)^3 \text{ EI}} (K_{12} + W \text{ Cos ak}) \qquad (48a)$$

and

$$B = \frac{-{L_i}^2}{(KL_i)^2 \text{ EI}} (K_{11} + \frac{LW}{KL_i} \text{Sin ak}) \qquad (48b)$$

By substituting these values of A and B into Equations (46) and (47) and rearranging, we get:

$$\frac{KLEI}{{L_i}^2} = (K_{11} + \frac{LW}{KL_i} \text{Sin ak}) \text{ Cos}(\frac{KL_i}{L_i} x) - \frac{L_i}{KL_i} (K_{12} + W \text{ Cos ak})$$

$$\text{Sin } (\frac{KL_i}{L_i} x) + (K_{12} + W)x - K_{11} - W_a \qquad a < x < L_i \qquad (49)$$

and

$$\frac{(KL_i)^2}{{L_i}^2} \text{EI } \frac{dy}{dx} = \frac{KL_i}{L_i} (K_{11} + \frac{LW}{KL_i} \text{Sin ak}) \text{ Sin}(\frac{KL_i}{L_i} x) - (K_{12} +$$

$$W \text{ Cos ak}) \; * \; \text{Cos } (\frac{KL_i}{L_i} x) + K_{12} + W \qquad a \leq x \leq L_i \qquad (50)$$

Finally, by knowing the boundary conditions $Y = 0$ and $\frac{dy}{dx} = 0$ at $x = L_i$, and eliminating K_{12} between Equations (49) and (50), the following expression for K_{11} is obtained:

$$K_{11} = -WL \frac{\frac{a}{L}(1 - \text{Cos } KL_i) - (1 - \text{Cos } ak)(\frac{\text{Sin } KL_i}{KL_i} - \text{Cos } KL_i)}{2(1 - \text{Cos } KL_i) - KL_i \text{ Sin } KL_i}$$

$$+ \frac{(\frac{1 - \text{Cos } KL_i}{KL_i} - \text{Sin } KL_i)(\text{Sin } ak)}{2(1 - \text{Cos } KL_i) - KL_i \text{ Sin } KL_i} \qquad (51)$$

and also knowing

$$\text{Cos } ak \text{ Cos } KL_i = \frac{1}{2}[\text{Cos } (ak + KL_i) + \text{Cos } (ak - KL_i)]$$

$$\text{Sin } ak \text{ Sin } KL_i = \frac{1}{2}[\text{Cos } (ak - KL_i) - \text{Cos } (ak + KL_i)]$$

and total length of member i is $L_i = a + b$, by substituting for $L_i - a = b$, Equation (51) will lead to:

$$K_{11} = \frac{-WL}{K}\left[\frac{\text{Sin } KL_i - \text{Sin } Ka - \text{Sin } Kb - Kb \text{ Cos } KL_i + KL_i \text{ Cos } Kb - Ka}{f_c}\right] \qquad (52)$$

For tensile axial load, Equation (52) may be expressed as:

$$K_{11} = -WL \left[\frac{\frac{a}{L}(1 - \text{Cosh } KL_i) - (1 - \text{Cosh } ak)(\frac{\text{Sinh } KL_i}{KL_i} - \text{Cosh } KL_i)}{2(1 - \text{Cosh } KL_i) + KL_i \text{ Sinh } KL_i}\right.$$

$$\left. + \frac{(\frac{1 - \text{Cosh } KL_i}{KL_i} + \text{Sinh } KL_i) \text{ Sinh } ak}{2(1 - \text{Cosh } KL_i) + KL_i \text{ Sinh } KL_i}\right] \qquad (53)$$

Equation (54) will reduce to:

$$K_{11} = \frac{-WL}{K}\left[\frac{\text{Sinh } KL_i - \text{Sinh } Ka - \text{Sinh } Kb - Kb \text{ Cosh } KL_i + KL_i \text{ Cosh } Kb - Ka}{2 - 2 \text{ Cosh } KL_i + KL_i \text{ Sinh } KL_i}\right] \quad (54)$$

The shear forces K_{12} and K_{22} can also be obtained by considering equilibrium conditions of the deformed beam element. Their values were shown in Table 2.

APPENDIX II

THE FINITE ELEMENT FORTRAN PROGRAM FOR THE SECOND-ORDER ANALYSIS OF RIGID FRAMES

This program will calculate the effect of secondary moment in cycle fashion and is written in Fortran language. The main program will initialize the Program's Parameters and then call on a set of six different subroutines. The flow chart of the Main Program is shown in Figure 5.

```
      PROGRAM FRAME(INPUT,OUTPUT,DATAF,OUTF,TAPE5=DATAF,TAPE6=OUTF)
C
C              PROGRAM 34 - MAIN PROGRAM
C
C  STATIC ANALYSIS FOR  PLANE FRAME SYSTEMS
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
C
      DIMENSION X(100),Y(100),CON(200),PROP(200),IB(80),TK(300,30),
     *AL(300),REAC(300),FORC(600),ELST(6,6),V(30),PF(600),SUM(200),
     *BL(300),SUE(200),PFORC(600)
C
C  INITIALIZATION OF PROGRAM PARAMETERS
C
C  NRMX  = ROW DIMENSION FOR THE STIFFNESS
C          MATRIX
C  NCMX  = COLUMN DIMENSION FOR THE STIFFNESS
C          MATRIX, OR MAXIMUM HALF BAND WIDTH
C          ALLOWED
C  NDF   = NUMBER OF DEGREES OF FREEDOM PER
C          NODE
C  NNE   = NUMBER OF NODES PER ELEMENT, EQUAL
C          TO 2 FOR BAR SYSTEMS
C  NDFEL = TOTAL NUMBER OF DEGREES OF FREEDOM
C          FOR ONE ELEMENT
C
      NRMX=300
      NCMX=30
      NDF=3
      NNE=2
      NDFEL=NDF*NNE
C
C  ASSIGN DATA SET NUMBERS TO IN, FOR INPUTT,
C  AND IO, FOR OUTPUT
C
      IN=5
      IO=6
      DO 1001 I=1,NE
      FORC((I-1)*6+1)=0
      PFORC((I-1)*6+1)=0
1001  PF((I-1)*6+1)=0
C
C APPLY THE ANALYSIS STEPS
      DO 310 IEND=1,11
      DO 410 IH=1,NE
      PFORC((IH-1)*6+1)=FORC((IH-1)*6+1)
410   CONTINUE
C
      DO 801 I1=1,NE
      DO 800 I2=2,6
??
```

```
      REAC((I1-1)*6+1)=0
      REAC((I1-1)*6+I2)=0
800   CONTINUE
801   CONTINUE
C  INPUTT
      CALL INPUTT(X,Y,CON,PROP,AL,IB,REAC,FORC,SUM,BL,SUE)
C  ASSEMBLING OF TOTAL STIFFNESS MATRIX
      PFORC((I-1)*6+1)=FORC((I-1)*6+1)
C
      CALL ASSEM(X,Y,CON,PROP.TK,ELST,AL,FORC,PFORC)
C
C  INTRODUCTION OF BOUNDARY CONDITIONS
C
      CALL BOUND(TK,AL,REAC,IB)
C
C  SOLUTION OF THE SYSTEM OF EQUATIONS
C
      CALL SLBSI(TK,AL,V,N,MS,NRMX,NCMX)
C
C  COMPUTATION OF MEMBER FORCES
C
      CALL FORCE(CON,PROP,FORC,REAC,X,Y,AL,PFORC)
C
C  OUTPUT
C
      CALL OUTPT(CON,AL,FORC,REAC)
C
      DO 1002 I=1,NE
      IF(ABS(FORC((I-1)*6+1)-PF((I-1)*6+1)).GE.0.0001) GO TO 1005
1002  CONTINUE
      STOP
1005  DO 1003 I=1,NE
1003  PF((I-1)*6+1)=FORC((I-1)*6+1)
310   CONTINUE
      END
      SUBROUTINE FEM (IEL,X,Y,CON,FORC,YZ,SUM,Q,IS,IE,SUE)
C
C
C  COMPUTATION OF FIXED-END ACTION
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION X(1),Y(1),CON(1),FORC(300),SUM(6),SUE(6)
C
C IEL = CURRENT ELEMENT
C  N1   = NUMBER OF START NODE
C  N2   = NUMBER OF END NODE
C
C PB  = BUCKLING LOAD
C SUM = END ACTION FOR START NODE
C SUE = END ACTION FOR END NODE
??
```

```
C SARM=  LENGTH OF PARTIAL ELEMENT
C A    = DISTANCE FROM CENTER OF ELEMENT TO START NODE
C B    = DISTANCE FROM CENTER OF ELEMENT TO END NODE
C
      L=NNE*(IEL-1)
      N1=CON(L+1)
      N2=CON(L+2)
      DX=X(N2)-X(N1)
      DY=Y(N2)-Y(N1)
      D=SQRT(DX**2+DY**2)
      SARM=D/50
      A=SARM/2
      B = D-A
      M = 3*NN
      DO 50 II=1,M
      SUM(II)=0
      SUE(II)=0
50    CONTINUE
      IF (FORC((IS-1)*6+1)) 101,111,121
C
C COMPUTE FIXED-END ACTION FOR TENSILE MEMBER
C
101   PB=((3.1415**2)*(E*YZ))/(D**2)
      VB=(-FORC((IS+1)*6+1)/PB)**.5
      BK=(-FORC((IS-1)*6+1)/(E*YZ))**.5
      FT=2-2*COSH(BK*D)+(BK*D)*SINH(BK*D)
      DO 103 KK=1,50
      JJ=NDF*(IS-1)
      SUM(JJ+1) = 0
      SUM(JJ+2)=SUM(JJ+2)+SARM*Q*((1/FT)*(COSH(BK*D)-COSH(BK*A)+COSH(BK*
     *B)+(BK*B)*SINH(BK*D)-1))
      SUM(JJ+3)=SUM(JJ+3)+SARM*Q*((1/(BK*FT))*(SINH(BK*D)-SINH(BK*A)-SIN
     *H(BK*B)-(BK*B)*COSH(BK*D)+(BK*D)*COSH(BK*B)-BK*A))
      JJ = NDF*(IE-1)
      SUE(JJ+1) = 0
      SUE(JJ+2)=SUE(JJ+2)+SARM*Q*((1/FT)*(COSH(BK*D)-COSH(BK*B)+COSH(BK
     **A)-(BK*A)*SINH(BK*D)-1))
      SUE(JJ+3)=SUE(JJ+3)+SARM*Q*((1/BK*FT)*(-SINH(BK*D)+SINH(BK*B)+SIN
     *H(BK*A)+(BK*A)*COSH(BK*D)-(BK*D)*COSH(BK*A)+(BK*B)))
      A = A+SARM
      B = D-A
103   CONTINUE
      RETURN
C
C COMPUTE FIXED-END ACTION FOR NONZERO FORCE MEMBER
C
111   DO 104 KK=1,50
      JJ=NDF*(IS-1)
      SUM(JJ+1) = 0
      SUM(JJ+2)=SUM(JJ+2)+SARM*Q*((-B**2)*(3*A+B)/D**3)
??
```

```
      SUM(JJ+3)=SUM(JJ+3)+SARM*O*(-A*(B**2)/D**2)
      JJ = NDF*(IE-1)
      SUE(JJ+1) = 0
      SUE(JJ+2)=SUE(JJ+2)+SARM*O*((-A**2)*(A+3*B)/D**3)
      SUE(JJ+3)=SUE(JJ+3)+SARM*O*((A**2)*B)/D**2
      A = A+SARM
      B = D-A
104   CONTINUE
      RETURN
C
C COMPUTE FIXED-END ACTION FOR COMPRESSION MEMBER
C
121   PB=((3.1415**2)*(E*YZ))/(D**2)
      VB=(FORC((IS-1)*6+1)/PB)**.5
      BK=(FORC((IS-1)*6+1)/(E*YZ))**.5
      FC=2-2*COS(BK*D)-(BK*D)*(SIN(BK*D))
      DO 105 KK=1,50
      JJ=NDF*(IS-1)
      SUM(JJ+1)=0
      SUM(JJ+2)=SUM(JJ+2)+SARM*O*((1/FC)*(COS(BK*D)-COS(BK*A)+COS(BK*B)+
     *(BK*B)*SIN(BK*D)-1))
      SUM(JJ+3)=SUM(JJ+3)+SARM*O*((1/(BK*FC))*(SIN(BK*D)-SIN(BK*A)-SIN(
     *BK*B)-(BK*B)*COS(BK*D)+(BK*D)*COS(BK*B)-BK*A))
      JJ = NDF*(IE-1)
      SUE(JJ+1) = 0
      SUE(JJ+2)=SUE(JJ+2)+SARM*O*((1/FC)*(COS(BK*D)-COS(BK*B)+COS(BK*A)
     *+(BK*A)*SIN(BK*D)-1))
      SUE(JJ+3)=SUE(JJ+3)+SARM*O*((1/(BK*FC))*(-SIN(BK*D)+SIN(BK*B)+SIN(
     *BK*A)+(BK*A)*COS(BK*D)-(BK*D)*COS(BK*A)+BK*B))
      A = A+SARM
C
C
C
      B = D-A
105   CONTINUE
      RETURN
      END
C
C             PROGRAM 35 - INPUTT PROGRAM
      SUBROUTINE INPUTT(X,Y,CON,PROP,AL,IB,REAC,FORC,SUM,BL,SUE)
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION X(1),Y(1),CON(1),PROP(1),AL(1),IB(1),REAC(1),W(3),IC(3),
     *FORC(600),SUM(300),BL(1),SUE(200)
C
C        ELEMENT PROPERTIES, NODAL LOADS, AND PRESCRIBED
C        UNKNOWN VALUES
C  IC = AUXILIARY ARRAY TO STORE TEMPORARELY THE CONNECTIVITY
C       OF AN ELEMENT, AND THE BOUNDARY UNKNOWNS STATUS
C       INDICATORS
??
```

```
C
C  READ BASIC PARAMETERS
C
C NN   = NUMBER OF NODES
C NE   = NUMBER OF ELEMENTS
C NLN = NUMBER OF LOADED NODES
C NBN = NUMBER OF BOUNDARY NODES
C NLE =NUMBER OF LOADED ELEMENT
C E    = MODULUS OF ELASTICITY
C
      WRITE(IO,20)
   20 FORMAT(" ",72("*"))
      READ(IN,1) NN,NE,NLN,NBN,E,NLE
      WRITE(IO,21) NN,NE,NLN,NBN,E,NLE
   21 FORMAT(//" INTERNAL DATA"//" NUMBER OF NODES            :",I5/" NUMBE
     *R OF ELEMENTS        :",I5/" NUMBER OF LOADED NODES  :",I5/" NUMBER
     *OF SUPPORT NODES :",I5/" MODULUS OF ELASTICITY    :",F10.0/" NUMBER
     * OF LOADED ELEMENTS:",I4////"NODAL COORDINATES"/7X,"NODE",6X,"X",
     *9X,"Y")
1     FORMAT(4I10,1F10.0,I10)
C
C  READ NODAL COORDINATES IN ARRAY X AND Y
C
      READ(IN,2) (I,X(I),Y(I), J=1,NN)
      WRITE(IO,2) (I,X(I),Y(I),I=1,NN)
    2 FORMAT(I10,2F10.4)
C
C  READ ELEMENT CONNECTIVITY IN ARRAY CON
C  AND ELEMENT PROPERTIES IN ARRAY PROP
C
      WRITE(IO,22)
   22 FORMAT(/" ELEMENT CONNECTIVITY AND PROPERTIES"/4X,"ELEMENT",3X,"ST
     *ART NODE  END NODE",5X,"AREA", 5X,"M. OF INERTIA")
      DO 3 I=1,NE
      READ(IN,4) ITEP,IC(1),IC(2),W(1),W(2)
      WRITE(IO,34) ITEP,IC(1),IC(2),W(1),W(2)
   34 FORMAT(3I10,2F15.5)
    4 FORMAT(3I10,2F10.5)
      N1=NNE*(I-1)
      PROP(N1+1)=W(1)
      PROP(N1+2)=W(2)
      CON(N1+1)=IC(1)
    3 CON(N1+2)=IC(2)
C
C  COMPUTE N, ACTUAL NUMBER OF UNKNOWNS, AND CLEAR THE LOAD
C  VECTOR
C
      WRITE(IO,23)
   23 FORMAT(/" NODAL LOADS"/7X,"NODE",5X,"PX",8X,"PY",8X,"MZ")
      N=NN*NDF
??
```

```
      DO 5 I=1,N
    5 BL(I)=0.
C
C  READ THE NODAL LOADS AND STORE THEM IN ARRAY AL
C
      IF (NLN.EQ. 0.) GO TO 166
      DO 6 I=1,NLN
      READ (IN,8) J,(W(K),K=1,NDF)
      WRITE(IO,8) J,(W(K),K=1,NDF)
    8 FORMAT(I10,3F10.2)
      DO 6 K=1,NDF
      L=NDF*(J-1)+K
      BL(L)=W(K)
6     CONTINUE
166   KKK=NN*NDF
      DO 180 II=1,KKK
      AL(II)=0
      SUE(II)=0
180   SUM(II)=0
C
      IF (NLE.EQ.0.) GO TO 80
      WRITE(IO,30)
      DO 44 MM=1,NLE
30    FORMAT(/"      NUMBER OF",5X,"START",3X,"END",3X,"UNIFORM"/1X,"LOADE
     *D ELEMENT",4X,"NODE",3X,"NODE",3X,"LOAD")
      READ (IN,88) IEL,IS,IE,Q
88    FORMAT(3I10,F15.5)
      WRITE(IO,99) IEL,IS,IE,Q
99    FORMAT(1I8,1I13,I7,F10.3)
      L3=NNE*(IEL-1)
      YZ=PROP(L3+2)
      CALL FEM (IEL,X,Y,CON,FORC,YZ,SUM,Q,IS,IE,SUE)
80    KZ=NN*NDF
      DO 156 IJ = 1,KZ
      AL(IJ)=AL(IJ)+BL(IJ)+SUM(IJ)+SUE(IJ)
156   CONTINUE
      IF (NLE.EQ.0.) GO TO 26
44    CONTINUE
C
C  READ BOUNDARY NODES DATA. STORE UNKNOWNS STATUS
C  INDICATORS IN ARRAY IB, AND PRESCRIBED UNKNOWNS
C  VALUES IN ARRAY REAC
C
26    WRITE(IO,24)
   24 FORMAT(/" BOUNDARY CONDITION DATA"/27X,"STATUS",24X,"PRESCRIBED VA
     *LUES"/19X,"(0:PRESCRIBED,  1:FREE)"/7X,"NODE",8X,"U",9X,"V",8X,"RZ"
     *,16X,"U",9X,"V",8X,"RZ")
      DO 7 I=1,NBN
      READ(IN,9) J,(IC(K),K=1,NDF),(W(K),K=1,NDF)
      WRITE(IO,10) J,(IC(K),K=1,NDF),(W(K),K=1,NDF)
??
```

```
      L1=(NDF+1)*(I-1)+1
      L2=NDF*(J-1)
      IB(L1)=J
      DO 7 K=1,NDF
      N1=L1+K
      N2=L2+K
      IB(N1)=IC(K)
    7 REAC(N2)=W(K)
    9 FORMAT(4I10,3F10.4)
   10 FORMAT(4I10,10X,3F10.4)
      RETURN
      END
C
      SUBROUTINE PDELTA(FORC,YZ,D,AK1,AK2,AK3,AK4,I)
C
C
C COMPUTATION OF COEF. USED IN STIFFNESS MATRIX
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION FORC(600)
C
C  YZ   = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
C PB  = BUCKLING LOAD
C AK1,AK2,AK3,AK4 ARE COEF USED IN STIFFNESS MATRIX
C
      IF(FORC((I-1)*6+1).LE.0.) GO TO 500
C
C COMPUTE THE COEF. FOR COMPRESSION MEMBER
C
      BK=(FORC((I-1)*6+1)/(E*YZ))**.5
      FC=2-2*COS(BK*D)-(BK*D)*(SIN(BK*D))
      AK1=(((BK*D)**3)*SIN(BK*D))/(12*FC)
      AK2=(((BK*D)**2)*(1-COS(BK*D)))/(6*FC)
      AK3=(BK*D)*(SIN(BK*D)-(BK*D)*COS(BK*D))/(4*FC)
      AK4=(BK*D)*((BK*D)-SIN(BK*D))/(2*FC)
      GO TO 1004
C
C COMPUTE THE COEF. FOR TENSION MEMBER
C
500   IF(FORC((I-1)*6+1).EQ.0.) GO TO 600
      BK=(-FORC((I-1)*6+1)/(E*YZ))**.5
      FT=2-2*COSH(BK*D)+(BK*D)*SINH(BK*D)
      AK1=(((BK*D)**3)*SINH(BK*D))/(12*FT)
      AK2=(((BK*D)**2)*(COSH(BK*D)-1))/(6*FT)
      AK3=(BK*D)*((BK*D)*COSH(BK*D)-SINH(BK*D))/(4*FT)
      AK4=(BK*D)*(SINH(BK*D)-(BK*D))/(2*FT)
      GO TO 1004
C
C COMPUTE THE COEF. FOR NONZERO MEMBER
C
??
```

```
600   AK1=1.0
      AK2=1.0
      AK3=1.0
      AK4=1.0
1004  CONTINUE
      RETURN
      END
      SUBROUTINE STIFF(NEL,X,Y,PROP,CON,ELST,AL,I,FORC)
C              PROGRAM 36
C  COMPUTATION OF ELEMENT STIFFNESS MATRIX FOR THE CURRENT ELEMENT
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION X(1),Y(1),CON(1),PROP(1),ELST(6,6),AL(1),
     *ROT(6,6),V(6),FORC(600)
C
C  NEL  = CURRENT ELEMENT NUMBER
C  N1   = NUMBER OF START NODE
C  N2   = NUMBER OF END NODE
C  AX   = AREA OF ELEMENT CROSS-SECTION
C  YZ   = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
C
      L=NNE*(NEL-1)
      N1=CON(L+1)
      N2=CON(L+2)
      AX=PROP(L+1)
      YZ=PROP(L+2)
C
C  COMPUTE LENGTH OF ELEMENT, AND SINE AND COSINE OF ITS LOCAL
C  X AXIS, AND STORE  IN D, SI, AND CO, RESPECTIVELY
C
      DX=X(N2)-X(N1)
      DY=Y(N2)-Y(N1)
      D=SQRT(DX**2+DY**2)
      CALL PDELTA(FORC,YZ,D,AK1,AK2,AK3,AK4,I)
      CO=DX/D
      SI=DY/D
      DO 1 I=1,6
      DO 1 J=1,6
      ELST(I,J)=0.
C
C  FORM ELEMENT ROTATION MATRIX
C
    1 ROT(I,J)=0.
      ROT(1,1)=CO
      ROT(1,2)=SI
      ROT(2,1)=-SI
      ROT(2,2)=CO
      ROT(3,3)=1.
      DO 2 I=1,3
      DO 2 J=1,3
```

??

```
    2 ROT(I+3,J+3)=ROT(I,J)
C
C   COMPUTE ELEMENT LOCAL STIFFNESS MATRIX
C
      ELST(1,1)=E*AX/D
      ELST(1,4)=-ELST(1,1)
      ELST(2,2)=(12*E*YZ/D**3)*AK1
      ELST(2,3)=(6*E*YZ/(D*D))*AK2
      ELST(2,5)=-ELST(2,2)
      ELST(2,6)=ELST(2,3)
      ELST(3,2)=ELST(2,3)
      ELST(3,3)=(4*E*YZ/D)*AK3
      ELST(3,5)=-ELST(2,3)
      ELST(3,6)=(2*E*YZ/D)*AK4
      ELST(4,1)=ELST(1,4)
      ELST(4,4)=ELST(1,1)
      ELST(5,2)=ELST(2,5)
      ELST(5,3)=ELST(3,5)
      ELST(5,5)=ELST(2,2)
      ELST(5,6)=ELST(3,5)
      ELST(6,2)=ELST(2,6)
      ELST(6,3)=ELST(3,6)
      ELST(6,5)=ELST(5,6)
      ELST(6,6)=ELST(3,3)
C
C   ROTATE ELEMENT STIFFNESS MATRIX TO GLOBAL COORDINATES
C
      CALL BTAB3(ELST,ROT,V,NDFEL,NDFEL)
      RETURN
      END
      SUBROUTINE FORCE(CON,PROP,FORC,REAC,X,Y,AL,PFORC)
C
C              PROGRAM 37
C   COMPUTATION OF ELEMENT FORCES
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION CON(1),PROP(1),FORC(1),REAC(1),X(1),Y(1),AL(1),ROT(3,3),
     *U(6),F(6),UL(6),FG(6),PFORC(600)
      DO 1 I=1,N
    1 REAC(I)=0.
      DO 100 NEL=1,NE
C
C   NEL  = CURRENT ELEMENT NUMBER
C   N1   = NUMBER OF START NODE
C   N2   = NUMBER OF END NODE
C   AX   = AREA OF ELEMENT CROSS-SECTION
C   YZ   = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
C
      L=NNE*(NEL-1)
      N1=CON(L+1)
```

??

```
      N2=CON(L+2)
      AX=PROF(L+1)
      YZ=PROF(L+2)
C
C   COMPUTE LENGTH OF ELEMENT, AND SINE AND COSINE OF ITS LOCAL
C   X AXIS, AND STORE  IN D, SI, AND CO, RESPECTIVELY
      DX=X(N2)-X(N1)
      DY=Y(N2)-Y(N1)
      D=SQRT(DX**2+DY**2)
C
      CO=DX/D
      SI=DY/D
C
C   FORM ELEMENT ROTATION MATRIX
C
      ROT(1,1)=CO
      ROT(1,2)=SI
      ROT(1,3)=0.
      ROT(2,1)=-SI
      ROT(2,2)=CO
      ROT(2,3)=0.
      ROT(3,1)=0.
      ROT(3,2)=0.
      ROT(3,3)=1.
C
C   ROTATE ELEMENT NODAL DISPLACEMENTS TO ELEMENT LOCAL
C   REFERENCE FRAME, AND STORE IN ARRAY UL
C
      K1=NDF*(N1-1)
      K2=NDF*(N2-1)
      DO 2 I=1,3
      J1=K1+I
      J2=K2+I
      U(I)=AL(J1)
    2 U(I+3)=AL(J2)
      DO 3 I=1,3
      UL(I)=0.
      UL(I+3)=0.
      DO 3 J=1,3
      UL(I)=UL(I)+ROT(I,J)*U(J)
    3 UL(I+3)=UL(I+3)+ROT(I,J)*U(J+3)
C
C   COMPUTE MEMBER END FORCES IN LOCAL COORDINATES
C
      IPP=NEL
      CALL PDELTA(PFORC,YZ,D,AK1,AK2,AK3,AK4,IPP)
      F(1)=E*AX/D*(UL(1)-UL(4))
      F(2)=12*E*YZ*AK1/(D**3)*(UL(2)-UL(5))+6*E*YZ*AK2/(D*D)*(
     1UL(3)+UL(6))
      F(3)=6*E*YZ*AK2/(D*D)*(UL(2)-UL(5))+2*E*YZ/D*(
??
```

```
     12*AK3*UL(3)+AK4*UL(6))
      F(6)=6*E*YZ*AK2/(D*D)*(UL(2)-UL(5))+2*E*YZ/D*(AK4*UL(3)+
     12*AK3*UL(6))
C  STORE MEMBER END FORCES IN ARRAY FORC
      F(4)=-F(1)
      F(5)=-F(2)
      I1=6*(NEL-1)
C
C
      DO 4 I=1,6
      I2=I1+I
    4 FORC(I2)=F(I)
C
C  ROTATE MEMBER END FORCES TO THE GLOBAL REFERENCE FRAME
C  AND STORE IN ARRAY FG
C
      DO 5 I=1,3
      FG(I)=0.
      FG(I+3)=0.
      DO 5 J=1,3
      FG(I)=FG(I)+ROT(J,I)*F(J)
    5 FG(I+3)=FG(I+3)+ROT(J,I)*F(J+3)
C
C  ADD ELEMENT CONTRIBUTION TO NODAL RESULTANTS,
C  IN ARRAY REAC
      DO 6 I=1,3
      J1=K1+I
      J2=K2+I
      REAC(J1)=REAC(J1)+FG(I)
    6 REAC(J2)=REAC(J2)+FG(I+3)
  100 CONTINUE
      RETURN
      END
      SUBROUTINE OUTPT(CON,AL,FORC,REAC)
C
C             PROGRAM 38 - OUTPUT PROGRAM
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION CON(1),AL(1),FORC(1),REAC(1)
C
C  WRITE NODAL DISPLACEMENTS
C
      WRITE(IO,1)
    1 FORMAT(//1X,72("*")//" RESULTS"//" NODAL DISPLACEMENTS"/7X,"NODE"
     *,11X,"U",14X,"V",13X,"RZ")
      DO 10 I=1,NN
      K1=NDF*(I-1)+1
      K2=K1+NDF-1
   10 WRITE(IO,2) I,(AL(J),J=K1,K2)
2     FORMAT(I10,6F15.6)
??
```

```
C
C   WRITE NODAL REACTIONS
C
      WRITE(IO,3)
    3 FORMAT(/" NODAL REACTIONS"/7X,"NODE",10X,"PX",13X,"PY",13X,"MZ")
      DO 20 I=1,NN
      K1=NDF*(I-1)+1
      K2=K1+NDF-1
   20 WRITE(IO,2) I,(REAC(J),J=K1,K2)
C
C   WRITE MEMBER END FORCES
C
      WRITE(IO,4)
    4 FORMAT(/" MEMBER FORCES"/6X,"MEMBER",5X,"NODE", 9X,"FX",13X,"FY",
     *13X,"MZ")
      DO 30 I=1,NE
      K1=6*(I-1)+1
      K2=K1+2
      N1=NNE*(I-1)
      ICON=CON(N1+1)
      WRITE(IO,6) I,ICON,(FORC(J),J=K1,K2)
      K1=K2+1
      K2=K1+2
      ICON=CON(N1+2)
   30 WRITE(IO,7) ICON,(FORC(J),J=K1,K2)
    6 FORMAT(2I10,3F15.4)
    7 FORMAT(I20,3F15.4)
      WRITE(IO,5)
    5 FORMAT(//1X,72("*"))
      RETURN
      END
      SUBROUTINE BOUND(TK,AL,REAC,IB)
C
C              PROGRAM 31
C
C   INTRODUCTION OF THE BOUNDARY CONDITIONS
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION AL(1),IB(1),REAC(1),TK(300,30)
      DO 100 L=1,NBN
C
C   NO =  NUMBER OF THE CURRENT BOUNDARY NODE
C
      L1=(NDF+1)*(L-1)+1
      NO=IB(L1)
      K1=NDF*(NO-1)
      DO 100 I=1,NDF
      L2=L1+I
      IF(IB(L2))100,10,100
C
??
```

```
C   PRESCRIBED UNKNOWN TO BE CONSIDERED
C   PLACE PRESCRIBED UNKNOWN VALUE IN AL
C
C   SET DIAGONAL COEFFICIENT OF TK EQUAL TO 1
C
   10 KR=K1+1
      DO 50 J=2,MS
      KV=KR+J-1
      IF(N-KV)30,20,20
C
C   MODIFY ROW OF TK AND CORRESPONDING ELEMENTS IN AL
C
   20 AL(KV)=AL(KV)-TK(KR,J)*REAC(KR)
      TK(KR,J)=0
   30 KV=KR-J+1
      IF(KV)50,50,40
C
C   MODIFY COLUMN IN TK AND CORRESPONDING ELEMENT  IN AL
C
   40 AL(KV)=AL(KV)-TK(KV,J)*REAC(KR)
      TK(KV,J)=0
   50 CONTINUE
      TK(KR,1)=1
      AL(KR)=REAC(KR)
  100 CONTINUE
      RETURN
      END
      SUBROUTINE ASSEM(X,Y,CON,PROP,TK,ELST,AL,FORC,PFORC)
C
C             PROGRAM 28
C   ASSEMBLING OF THE TOTAL MATRIX FOR THE PROBLEM
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION X(1),Y(1),CON(1),TK(300,30),ELST(6,6),
     *PROP(1),AL(1),FORC(600),PFORC(600)
C
C   COMPUTE HALF BAND WIDTH AND STORE IN MS
C
      N1=NNE-1
      MS=0
      DO 7 I=1,NE
      L1=NNE*(I-1)
      DO 7 J=1,N1
      L2=L1+J
      J1=J+1
      DO  7 K=J1,NNE
      L3=L1+K
      L=ABS(CON(L2)-CON(L3))
      IF(MS-L)6,7,7
    6 MS=L
??
```

```
    7 CONTINUE
      MS=NDF*(MS+1)
C
C  CLEAR THE TOTAL STIFFNESS MATRIX
C
      DO 10 I=1,N
      DO 10 J=1,MS
   10 TM(I,J)=0
C
C  LOOP ON THE ELEMENTS AND ASSEMBLE THE TOTAL STIFFNESS MATRIX
C
      DO 20 NEL=1,NE
      I=NEL
C
C  STIFF COMPUTES THE STIFFNESS MATRIX FOR ELEMENT NEL
      CALL STIFF(NEL,X,Y,PROP,CON,ELST,AL,I,FORC)
C
C
C  ELASS PLACES THE STIFFNESS MATRIX OF ELEMENT NEL IN THE TOTAL
C  STIFFNESS MATRIX
C
   20 CALL ELASS(NEL,CON,TM,ELST)
      RETURN
      END
      SUBROUTINE ELASS(NEL,CON,TM,ELMAT)
C
C             PROGRAM 30
C
C  THIS PROGRAM STORES THE ELEMENT MATRIX FOR ELEMENT NEL IN
C  THE TOTAL MATRIX FOR THE PROBLEM
C
      COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,N,MS,IN,IO,E,G,NLE
      DIMENSION CON(1),TM(300,30),ELMAT(6,6)
C
C  NEL  = CURRENT ELEMENT NUMBER
C  N1   = NUMBER OF START NODE
C  N2   = NUMBER OF END NODE
C
      L1=NNE*(NEL-1)
      DO  50 I=1,NNE
      L2=L1+I
      N1=CON(L2)
      I1=NDF*(I-1)
      J1=NDF*(N1-1)
      DO  50 J=I,NNE
      L2=L1+J
      N2=CON(L2)
      I2=NDF*(J-1)
      J2=NDF*(N2-1)
      DO 50 K=1,NDF
??
```

```
      K1=1
      IF(N1-N2)20,10,30
C
C  STORE A DIAGONAL SUBMATRIX
C
   10 KI=K
C
C  STORE AN OFF DIAGONAL SUBMATRIX
C
   20 KR=J1+K
      IC=J2-KR+1
      K1=I1+K
      GO TO 40
C
C  STORE THE TRANSPOSE OF AN OFF DIAGONAL MATRIX
C
   30 KR=J2+K
      IC=J1-KR+1
      K2=I2+K
   40 DO 50 L=KI,NDF
      KC=IC+L
      IF(N1-N2)45,45,46
   45 K2=I2+L
      GO TO 50
   46 K1=I1+L
   50 TM(KR,KC)=TM(KR,KC)+ELMAT(K1,K2)
      RETURN
      END
      SUBROUTINE SLBSI(A,B,D,N,MS,NX,MX)
C
C PROGRAM 15
C
C SOLUTION OF LINEAR SYSTEM OF EQUATIONS
C BY THE GAUSS ELIMINATION METHOD,FOR
C SYMMETRIC BANDED SYSTEMS
C
C A : ARRAY CONTAINING THE UPPER TRIANGULAR
C     PART OF THE SYSTEM MATRIX, STORED
C     ACCORDING TO THE SYMMETRIC BANDED
C     SCHEME
C B : ORIGINALLY IT CONTAINS THE INDEPENDENT
C     COEFFICIENTS. AFTER SOLUTION IT CONTAINS
C     THE VALUES OF THE SYSTEM UNKNOWNS.
C
C N : ACTUAL NUMBER OF UNKNOWNS
C N : ACTUAL NUMBER OF UNKNOWNS
C MS: ACTUAL HALF BANDWIDTH
C NX: ROW DIMENSION OF A AND B
C MX: COLUMN DIMENSION OF A
C
??
```

```
  200 B(K)=B(I )-A(K,K2)*B(J)
  300 RETURN
      END
      SUBROUTINE BTAB3(A,B,V,N,NX)
C
C PROGRAM 9
C
C THIS PROGRAM COMPUTES THE MATRIX
C OPERATION A = TRANSPOSE(B) * A * B,
C WHERE A AND B ARE SQUARE MATRICES
C N : ACTUAL ORDER OF A AND B
C NX: ROW AND COLUMN DIMENSION OF A AND B
C
C V : AUXILIARY VECTOR
C
      DIMENSION A(NX,NX),B(NX,NX),V(NX)
C
C COMPUTE A * B AND STORE IN A
C
      DO 10 I=1,N
      DO  5 J=1,N
      V(J)=0.
      DO 5 K=1,N
    5 V(J)=V(J)+A(I,K)*B(K,J)
      DO 10 J=1,N
   10 A(I,J)=V(J)
C
C COMPUTE TRANSPOSE(B) * A AND STORE IN A
C
      DO 20 J=1,N
      DO 15 I=1,N
      V(I)=0.
      DO 15 K=1,N
   15 V(I)=V(I)+B(K,I)*A(K,J)
      DO 20 I=1,N
   20 A(I,J)=V(I)
      RETURN
      END
--EOR--
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??
```

APPENDIX III

INPUT AND OUTPUT RESULTS OF FIRST-ORDER AND THREE CYCLES OF SECOND-ORDER ANALYSIS FOR EXAMPLE 1

INTERNAL DATA

```
NUMBER OF NODES              :   5
NUMBER OF ELEMENTS           :   4
NUMBER OF LOADED NODES       :   2
NUMBER OF SUPPORT NODES      :   2
MODULUS OF ELASTICITY        :   29000.
NUMBER OF LOADED ELEMENTS:       0
```

NODAL COORDINATES

NODE	X	Y
1	0.0000	0.0000
2	0.0000	120.0000
3	60.0000	120.0000
4	180.0000	120.0000
5	180.0000	-120.0000

.EMENT CONNECTIVITY AND PROPERTIES

ELEMENT	START NODE	END NODE	AREA	M. OF INERTIA
1	1	2	3.17400	7.23300
2	2	3	3.17400	7.23300
3	3	4	3.17400	7.23300
4	4	5	3.17400	7.23300

NODAL LOADS

NODE	FX	FY	MZ
2	5.00	0.00	0.00
3	0.00	-20.00	0.00

BOUNDARY CONDITION DATA

	STATUS (O:PRESCRIBED, 1:FREE)			PRESCRIBED VALUES		
NODE	U	V	RZ	U	V	RZ
1	0	0	0	0.0000	0.0000	0.0[illegible]
5	0	0	0	0.0000	0.0000	0.0[illegible]

****

RESULTS

NODAL DISPLACEMENTS

NODE	U	V	RZ
1	0.000000	0.000000	0.000000
2	8.963128	-.015354	-.121699
3	8.961446	-5.875502	-.039951

??

4	8.958083	-.021439	.043438
5	0.000000	0.000000	0.000000

NODAL REACTIONS

NODE	FX	FY	MZ
1	-2.419797	11.777498	357.913961
2	5.000000	-.000000	-.000000
3	.000000	-20.000000	-.000000
4	.000000	-.000000	.000000
5	-2.580203	8.222502	271.659980

MEMBER FORCES

MEMBER	NODE	FX	FY	MZ
1	1	11.7775	2.4198	357.9140
	2	-11.7775	-2.4198	-67.5383
2	2	2.5802	11.7775	67.5383
	3	-2.5802	-11.7775	639.1116
3	3	2.5802	-8.2225	-639.1116
	4	-2.5802	8.2225	-347.5887
4	4	8.2225	2.5802	347.5887
	5	-8.2225	-2.5802	271.6600

****
****

RESULTS

NODAL DISPLACEMENTS

NODE	U	V	RZ
1	0.000000	0.000000	0.000000
2	10.765617	-.014734	-.138186
3	10.763987	-6.409525	-.041834
4	10.760726	-.022679	.046890
5	0.000000	0.000000	0.000000

NODAL REACTIONS

NODE	FX	FY	MZ
1	-2.498966	11.302028	438.380934
2	5.000000	-.000000	0.000000
3	.000000	-20.000000	-.000000
4	.000000	-.000000	0.000000
5	-2.501034	8.697972	311.400794

MEMBER FORCES

MEMBER	NODE	FX	FY	MZ
1	1	11.3020	2.4990	438.3809
	2	-11.3020	-2.4990	-11.7130

??

2	2	2.5010	11.3020	11.7130
	3	-2.5010	-11.3020	682.9086
3	3	2.5010	-8.6980	-682.9086
	4	-2.5010	8.6980	-377.3274
4	4	8.6980	2.5010	377.3274
	5	-8.6980	-2.5010	311.4008

****
****

RESULTS

NODAL DISPLACEMENTS

NODE	U	V	RZ
1	0.000000	0.000000	0.000000
2	10.736374	-.014754	-.137908
3	10.734763	-6.402321	-.041854
4	10.731540	-.022640	.047020
5	0.000000	0.000000	0.000000

NODAL REACTIONS

NODE	FY	FY	MZ
1	-2.528305	11.316851	437.637274
2	5.000000	-.000000	.000000
3	.000000	-20.000000	.000000
4	.000000	-.000000	.000000
5	-2.471695	8.683149	310.704488

MEMBER FORCES

MEMBER	NODE	FX	FY	MZ
1	1	11.3169	2.5283	437.6373
	2	-11.3169	-2.5283	-12.8979
2	2	2.4717	11.3169	12.8979
	3	-2.4717	-11.3169	682.0887
3	3	2.4717	-8.6831	-682.0887
	4	-2.4717	8.6831	-375.8450
4	4	8.6831	2.4717	375.8450
	5	-8.6831	-2.4717	310.7045

****
****

RESULTS

??

NODAL DISPLACEMENTS

NODE	U	V	RZ
1	0.000000	0.000000	0.000000
2	10.735890	-.014753	-.137888
3	10.734278	-6.400970	-.041839
4	10.731055	-.022641	.047000
5	0.000000	0.000000	0.000000

NODAL REACTIONS

NODE	FX	FY	MZ
1	-2.527932	11.316458	437.639243
2	5.000000	-.000000	.000000
3	.000000	-20.000000	.000000
4	.000000	-.000000	.000000
5	-2.472068	8.683542	310.666675

MEMBER FORCES

MEMBER	NODE	FX	FY	MZ
1	1	11.3165	2.5279	437.6392
	2	-11.3165	-2.5279	-12.7909
2	2	2.4721	11.3165	12.7909
	3	-2.4721	-11.3165	681.9814
3	3	2.4721	-8.6835	-681.9814
	4	-2.4721	8.6835	-375.8089
4	4	8.6835	2.4721	375.8089
	5	-8.6835	-2.4721	310.6667

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**************************************************************************
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